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Diffraction of axisymmetric waves in a borehole by bed boundary discontinuities

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ABSTRACT

This paper presents the calculation of the diffraction of axisymmetric borehole waves by bed boundary discontinuities. The bed boundary is assumed to be horizontal and the inhomogeneities to be axially symmetric. In such a geometry, an axially symmetric source will produce only axially symmetric waves. Since the borehole is an open structure, the mode spectrum consists of a discrete part as well as a continuum. The scattering of a continuum of waves by bed boundaries is difficult to treat. The approach used in the past in treating this class of problem has been approximate in nature, or highly numerical, such as the finite-element method. We present here a systematic way to approximate the continuum of modes by discrete modes. After discretization, the scattering problem can be treated simply. Since the approach is systematic, it allows derivation of the solution to any desired degree of accuracy in theory; but in practice, it is limited by the computational resources available. We also show that our approach is variational and satisfies both the reciprocity theorem and energy conservation.

INTRODUCTION

The scattering of electromagnetic (EM) waves by horizontal bed boundaries is an important problem in borehole geophysical probing. Bed boundaries change the response of tools, making the interpretation of logs difficult. With a good understanding of the diffraction of waves by a bed boundary, we may be able to eliminate the effect of the bed boundary on tool measurements.

A borehole passing through a bed boundary can be viewed as the junction of two open waveguides. For any wave incident on the bed boundary, the reflected and transmitted waves are determined by the requirement that the tangential electric and magnetic fields are continuous across the plane of the bed boundary. This condition can be phrased as an integral equation for the field in the plane of the bed boundary. We extracted useful information from this equation by variational techniques (Angulo, 1957; Angulo and Chang, 1959; Bresler, 1959; Reinhart et al., 1971; Hockham and Sharpe, 1972; Ikegami, 1972; Rozzi, 1978; Rozzi and Veld, 1980), Wiener-Hopf methods (Angulo and Chang, 1953; Angulo and Chang, 1959; Kay, 1959; Aoki and Miyazaki, 1982), expansion in Neuman series (Gelin et al., 1979; Gelin et al., 1981), and mode matching (Clarricoats et al., 1972; Hockham and Sharpe, 1972; Brooke and Kharadly, 1982; Mahmoud and Beal, 1975; Morishita et al., 1979). The most significant technical complication of junction problems in an open waveguide is the treatment of the continuous spectrum. Pudensi and Ferreira (1982) showed how the continuum modes (also known as radiation modes) can be systematically approximated by a set of discrete modes using Hermite functions, which form a complete set on an infinite interval. In this work, this technique is generalized to cylindrical structures and developed further by using more general, possibly nonorthogonal expansion functions. We obtain the discrete modes by solving the differential equation first, obviating the need to formulate and solve an integral equation. It applies to an inhomogeneous waveguide of arbitrary profile. We relate our technique to other variational methods (Strang and Fix, 1973; Harrington, 1968). In the Appendix we show that the result is energy-conserving and that it satisfies reciprocity. We consider only the diffraction of axially symmetric waves here. The diffraction of nonaxisymmetric waves will be a topic of future research.

The geometric configuration we consider is shown in Figure 1. The region $\rho < b$ represents a fluid-filled borehole. The regions $\varepsilon_i(\rho)$, $\mu_i(\rho)$ where $i = 1, 2$ and $\rho > b$ represent invaded zones in the rock formation that are flushed out by the drilling fluid, reflecting zones of altered electrical properties. Regions I and II are two different beds in the geologic formation being probed. The current loop can be either a magnetic current or an electric current loop since the two problems are duals of each other. The problem under consideration is axially symmetric, and hence two-dimensional (2-D). We subsequently show how this problem is reduced to a one-dimensional (1-D) problem. The method compares very favorably with the finite-element method (FEM) (Anderson and Chang, 1982; Anderson and

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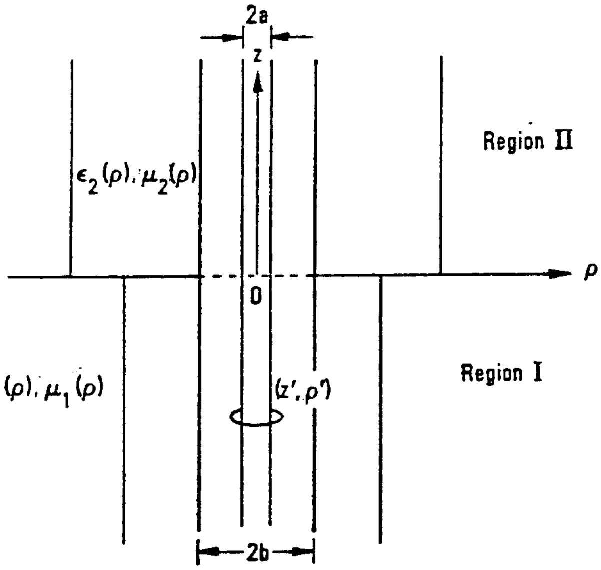

FIG. 1. Theoretical model of a borehole with a horizontal bed boundary.

Chang, 1983) in computation time when the FEM is used to solve problems of this type. This is because the FEM essentially treats the problem as two-dimensional. Our approach finds the modes of the structure by solving a 1-D differential equation. We circumvent the need to solve a surface integral equation which usually is singular. Our approach is directly applicable to the modeling of the EM wave propagation on drill stems passing through beds (Wait and Hill, 1979; Bangdowan and Trofimenloff, 1982), as well as the dielectric logging tool (Huchital et al., 1982; Chew, 1982; Chew, 1984) and the induction tools (Moran and Kunz, 1962). It can also easily be generalized to assess the effect of bed boundaries on other mandrel tools involving wave propagation phenomena.

FORMULATION

A vertical magnetic dipole can be formed by a horizontal electric current loop, while a vertical electric dipole can be formed by a horizontal magnetic current loop. We can construct a magnetic current loop from a toroidal antenna carrying electric current since magnetic charges or currents do not exist. Hence, we consider only the radiation of electric or magnetic current loops.

From Maxwell's equations, we show that if we have magnetic current only or electric current only, the vector wave equations satisfied by the magnetic field or the electric field are, respectively [assuming $\exp(-i\omega t)$ time dependence],

$$\varepsilon\nabla \times \varepsilon^{-1}\nabla \times \mathbf{H} - \omega^2\mu\varepsilon\mathbf{H} = i\omega\varepsilon\mathbf{J}_m, \tag{1}$$

and

$$\mu\nabla \times \mu^{-1}\nabla \times \mathbf{E} - \omega^2\mu\varepsilon\mathbf{E} = i\omega\mu\mathbf{J}_e. \tag{2}$$

In the above, ε and μ are functions of $\mathbf{r}$, indicating an electrically inhomogeneous medium. Equations (1) and (2) are also duals of each other. When $\mathbf{J}_m$ and $\mathbf{J}_e$ have only a ϕ-component, are axially symmetric, and μ and ε are independent of ϕ (i.e., axially symmetric inhomogeneities), $\mathbf{H}$ in equation (1) and $\mathbf{E}$ in equation (2) contain only $\hat{\phi}$-components. They are, respectively, known as transverse magnetic (TM) waves and transverse electric (TE) waves. Under such assumption of axisymmetry, equations (1) and (2) simplify to

$$\left[\rho p \frac{\partial}{\partial \rho} \frac{1}{\rho p} \frac{\partial}{\partial \rho} + p \frac{\partial}{\partial z} \frac{1}{p} \frac{\partial}{\partial z} + \omega^2\mu\varepsilon\right]\rho A_\phi = -i\omega p I \rho\delta(\rho - \rho')\delta(z - z'). \tag{3}$$

In the above equation A_ϕ is H_ϕ or E_ϕ depending upon whether it describes a TM or TE wave. We assume that the source is a current loop of radius ρ' located at z'; p and I are, respectively, ε and *magnetic current* for the TM wave whereas they are, respectively, μ and *electric current* for the TE wave.

For the geometry of Figure 1, we assume that ε and μ are arbitrary functions of ρ, while they are independent of z in each region. Hence, it is more expedient to find eigenfunctions which are the homogeneous solutions to equation (3) that satisfy

$$\left[\rho p \frac{\partial}{\partial \rho} \frac{1}{\rho p} \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} + \omega^2\mu\varepsilon\right] F(\rho, z) = 0. \tag{4}$$

To find the solutions to equation (4), we apply the idea of separation of variables and assume that

$$F(\rho, z) = \sum_\alpha f_\alpha(\rho) e^{ik_{\alpha z} z} a_\alpha, \tag{5}$$

where a_α is a constant independent of ρ and z. It follows that

$$\left[\rho p \frac{\partial}{\partial \rho} \frac{1}{\rho p} \frac{\partial}{\partial \rho} + \omega^2\mu\varepsilon - k_{\alpha z}^2\right] f_\alpha(\rho) = 0, \tag{6}$$

which is a 1-D equation to be solved numerically. The boundary conditions to be satisfied by $f_\alpha(\rho)$ are that

$$f_\alpha(\rho) = 0, \qquad \rho \to \infty,$$

and (7)

$$\frac{1}{\rho} f_\alpha(\rho) = 0, \qquad \rho = 0.$$

If there is a center metallic pipe of radius a in the borehole, the boundary condition requires that the tangential electric field be zero at the pipe surface. Hence, for TM waves, the boundary condition is

$$\left.\frac{\partial}{\partial \rho} f_\alpha(\rho)\right|_{\rho = a} = 0, \tag{8a}$$

while for TE waves, the boundary condition is

$$\left.\frac{1}{\rho} f_\alpha(\rho)\right|_{\rho = a} = 0. \tag{8b}$$

Notice that in the presence of a metallic pipe, the boundary condition for the TM waves at the borehole center is drastically changed. Hence, the presence of a center pipe in a borehole is a singular perturbation for TM waves, drastically altering the physical characteristics of the waves.

To obtain the solution to equation (6), we attempt a numerical scheme, so that solutions can be sought for μ and ε which are arbitrary functions of ρ. First, we choose a set of basis functions which is complete over the interval $(a, +\infty)$, so that we can expand

$$f_\alpha(\rho) = \sum_{n=1}^{\infty} b_{\alpha n} g_n(\rho). \tag{9}$$

Depending upon whether we are dealing with the TM case or the TE case, the boundary conditions on $g_n(\rho)$ on the center pipe surface are

$$\frac{\partial}{\partial \rho} g_n(\rho)\bigg|_{\rho=a} = 0, \qquad \text{TM case,} \tag{10a}$$

and

$$\frac{1}{\rho} g_n(\rho)\bigg|_{\rho=a} = 0, \qquad \text{TE case.} \tag{10b}$$

The boundary conditions of equation (10) imply the boundary conditions in equations (8a) and (8b) if we are working with a finite summation in equation (9). Substituting the first N terms of equation (9) into equation (6), we obtain

$$\sum_{n=1}^{N} b_{\alpha n}\left[\rho p \frac{\partial}{\partial \rho}\frac{1}{\rho p}\frac{\partial}{\partial \rho} + \omega^2 \mu\varepsilon - k_{\alpha z}^2\right] g_n(\rho) = 0. \tag{11}$$

We multiply the above by $(1/\rho p) g_m(\rho)$ and integrate from a to ∞ to eliminate the ρ dependence in equation (11).

Defining the inner product as

$$\langle f, g\rangle = \int_a^{\infty} d\rho \frac{1}{\rho p} f(\rho) g(\rho), \tag{12}$$

equation (11) becomes

$$\sum_{n=1}^{N} b_{\alpha n}[B_{nm} - k_{\alpha z}^2 G_{nm}] = 0, \tag{13}$$

where

$$B_{nm} = \left\langle g_m, \rho p \frac{\partial}{\partial \rho}\frac{1}{\rho p}\frac{\partial}{\partial \rho} g_n\right\rangle + \omega^2 \langle g_m, \mu\varepsilon g_n\rangle, \tag{13a}$$

and

$$G_{nm} = \langle g_m, g_n\rangle. \tag{13b}$$

Using integration by parts, we can show that

$$B_{nm} = -\int_a^{\infty} d\rho \frac{1}{\rho p} g_n'(\rho) g_m'(\rho) + \omega^2 \int_a^{\infty} d\rho \frac{\mu\varepsilon}{\rho p} g_n(\rho) g_m(\rho), \tag{14}$$

where the primes indicate derivatives with respect to the argument of the functions. Hence, with the definition of the inner product in equation (12), B_{nm} and G_{nm} are symmetric matrices. They are also the matrix representations of the differential operator in equation (6). The symmetry of B_{nm} and G_{nm} is the consequence of the self-adjointness or reciprocal nature of the problem. Therefore, g_m and g_n are not necessarily orthogonal to each other, since p in equation (12) is an arbitrary function of ρ.

The eigenvalues $k_{\alpha z}^2$ in equation (13) are obtained by solving

$$[\underset{\sim}{G}^{-1} \cdot \underset{\sim}{B} - k_{\alpha z}^2 \underset{\sim}{I}]\mathbf{b}_\alpha = 0. \tag{15}$$

Because of the symmetry of the $\underset{\sim}{G}$ and $\underset{\sim}{B}$ matrices, the eigenvalues thus obtained are variationally good. Also, if $\underset{\sim}{G}$ and $\underset{\sim}{B}$ are Hermitian, the eigenvalues $k_{\alpha z}^2$ are all real. This corresponds to the case of lossless media. We show in the Appendix that the procedure is also equivalent to solving for the eigenvalues of equation (6) using the Rayleigh-Ritz procedure, which is a well-known variational method. The eigenvectors can be obtained from equation (15), which can be used to derive the eigenfunctions in equation (9). If N basis functions are used in equation (9), equation (15) will produce N eigenvalues and N eigenvectors. Hence, there are N eigenfunctions available.

Eigenvectors obtained by solving equation (13) satisfy $\underset{\sim}{G}$ orthogonality, i.e.,

$$\mathbf{b}_\alpha^t \underset{\sim}{G} \mathbf{b}_\beta = \delta_{\alpha\beta} D_\alpha. \tag{16}$$

Because of this, we can prove that the eigenfunctions are orthogonal with the inner product defined in equation (12), i.e.,

$$\int_a^{\infty} \frac{d\rho}{\rho p} f_\alpha(\rho) f_\beta(\rho) = \sum_{n=1}^{N}\sum_{m=1}^{N} b_{\alpha n} b_{m\beta}^t \int_a^{\infty} d\rho \frac{1}{\rho p} g_n(\rho) g_m(\rho)$$
$$= \sum_{n=1}^{N}\sum_{m=1}^{N} b_{\alpha n} G_{nm} b_{m\beta}^t = \delta_{\alpha\beta} D_\alpha. \tag{17}$$

We can define the eigenfunctions to be orthonormal but we leave them as they are.

With the availability of the eigenfunctions and the eigenvalues, we seek an approximate solution to equation (3) in a systematic fashion. First, we assume p to be independent of z, i.e., the bed boundaries are absent. In such a case, we can expand the field in terms of the eigenfunctions as

$$\rho A_\phi = \sum_{\alpha=1}^{N} a_\alpha f_\alpha(\rho), \tag{18}$$

where a_α is a function of z. Substituting into equation (3), we have

$$\sum_{\alpha=1}^{N}\left(\frac{\partial^2}{\partial z^2} + \rho p \frac{\partial}{\partial \rho}\frac{1}{\rho p}\frac{\partial}{\partial \rho} + k^2(\rho)\right) a_\alpha f_\alpha(\rho) = -i\omega p I \rho' \delta(\rho - \rho')\delta(z - z'). \tag{19}$$

Multiplying the above by $(1/\rho p) f_\beta(\rho)$ and integrating from a to ∞, we get

$$\sum_{\alpha=1}^{N} \frac{\partial^2}{\partial z^2}\left[a_\alpha \int_a^{\infty} \frac{d\rho}{\rho p} f_\beta(\rho) f_\alpha(\rho)\right] + a_\alpha\left[\int_a^{\infty} \frac{d\rho}{\rho p} f_\beta(\rho)\left(\rho p \frac{\partial}{\partial \rho}\frac{1}{\rho p}\frac{\partial}{\partial \rho} + k^2\right) f_\alpha(\rho)\right] = -i\omega I \delta(z - z') f_\beta(\rho'). \tag{20}$$

However,

$$\int_a^{\infty} \frac{d\rho}{\rho b} f_\beta(\rho)\left(\rho p \frac{\partial}{\partial \rho}\frac{1}{\rho p}\frac{\partial}{\partial \rho} + k^2\right) f_\alpha(\rho) = k_{\alpha z}^2 \int_a^{\infty} \frac{d\rho}{\rho p} f_\alpha(\rho) f_\beta(\rho) = k_{\alpha z}^2 \delta_{\alpha\beta} D_\alpha. \tag{21}$$

Hence, equation (20) becomes

$$\left[\frac{\partial^2}{\partial z^2} + k_{\beta z}^2\right] a_\beta = -i\omega I \delta(z - z') f_\beta(\rho') D_\beta^{-1}. \tag{22}$$

Solving equation (22), we find that

$$a_\beta = -\frac{\omega I f_\beta(\rho')}{2k_{\beta z}D_\beta} e^{ik_{\beta z}|z-z'|}. \tag{23}$$

Hence, the approximate solution to equation (3) is

$$\rho A_\phi = -\frac{\omega I}{2}\sum_{\alpha=1}^{N}\frac{f_\alpha(\rho')f_\alpha(\rho)}{k_{\alpha z}D_\alpha} e^{ik_{\alpha z}|z-z'|}. \tag{24}$$

The accuracy of the solution can be increased systematically if we take more terms in the expansion of equation (9), in other words, make N in equation (24) larger. Equation (24) is the general soluton for the case with a center conducting pipe. When the pipe is absent, the boundary condition for the TE case is still the same as given by Equation (8a), by setting $a = 0$. For TM waves, the presence of a pipe represents a singular perturbation. Its effect is still felt no matter how small a is. But the TM case in the absence of a pipe is the dual of the TE case in the absence of a pipe. Hence, the solution can be derived from equation (24).

BED BOUNDARY REFLECTION

In the presence of a bed boundary at $z = 0$, we write the field below the bed boundary as the superposition of field due to the primary source excitation [equation (24)] and reflection by the bed boundary, i.e.,

$$\rho A_{1\phi} = -\frac{\omega I}{2}\sum_{\alpha=1}^{N} f_{1\alpha}(\rho)\left[\frac{f_{1\alpha}(\rho')}{k_{1\alpha z}D_{1\alpha}} e^{ik_{1\alpha z}|z-z'|} + \left(\sum_{\gamma=1}^{N} R_{\alpha\gamma}\frac{f_{1\gamma}(\rho')}{k_{1\gamma z}D_{1\gamma}} e^{ik_{1\gamma z}|z'|}\right)e^{ik_{1\alpha z}z}\right]. \tag{25}$$

The field above the bed boundary, or in region II, can be written as

$$\rho A_{2\phi} = -\frac{\omega I}{2}\sum_{\alpha=1}^{N} f_{2\alpha}(\rho) \times\left(\sum_{\gamma=1}^{N} T_{\alpha\gamma}\frac{f_{1\gamma}(\rho')}{k_{1\gamma z}D_{1\gamma}} e^{ik_{1\gamma z}|z'|}\right)e^{ik_{2\alpha z}z}. \tag{26}$$

The radial component of the field can be derived by the equation

$$A_\rho = \pm\frac{1}{i\omega p}\frac{\partial}{\partial z}A_\phi, \tag{27}$$

where the " + " sign is chosen for TM waves while the " − " sign is chosen for TE waves. With equation (27), we derive that

$$\pm\rho A_{1\rho} = -\frac{I}{2p_1(\rho)}\sum_{\alpha=1}^{N} k_{1\alpha z}f_{1\alpha}(\rho)\left\{\pm\frac{f_{1\alpha}(\rho')}{k_{1\alpha z}D_{1\alpha}} e^{-ik_{1\alpha z}|z-z'|} - \left[\sum_{\gamma=1}^{N} R_{\alpha\gamma}\frac{f_{1\gamma}(\rho')}{k_{1\gamma z}D_{1\gamma}} e^{ik_{1\gamma z}|z'|}\right]e^{-ik_{1\alpha z}z}\right\}, \tag{28}$$

and

$$\pm\rho A_{2\rho} = -\frac{I}{2p_2(\rho)}\sum_{\alpha=1}^{N} k_{2\alpha z}f_{2\alpha}(\rho) \times\left[\sum_{\gamma=1}^{N} T_{\alpha\gamma}\frac{f_{1\gamma}(\rho')}{k_{1\gamma z}D_{1\gamma}} e^{ik_{1\gamma z}|z'|}\right]e^{ik_{2\alpha z}}. \tag{29}$$

The expressions in brackets in equations (25)–(29) are defined so as to simplify the derivation of $R_{\alpha\gamma}$ and $T_{\alpha\gamma}$ later. We will see easily that equations (25)–(29) can be written more concisely in vector notation as

$$\rho A_{1\phi} = -\frac{\omega I}{2}\mathbf{f}_1^t(\rho)[e^{i\underline{\mathbf{K}}_{1z}|z-z'|} + e^{-i\underline{\mathbf{K}}_{1z}z}\underline{\mathbf{R}}_{12}e^{i\underline{\mathbf{K}}_{1z}|z'|}] \times \underline{\mathbf{K}}_{1z}^{-1}\underline{\mathbf{D}}_1^{-1}\mathbf{f}_1(\rho'), \tag{30}$$

$$\rho A_{2\phi} = -\frac{\omega I}{2}\mathbf{f}_2^t(\rho)e^{i\underline{\mathbf{K}}_{2z}z}\underline{\mathbf{T}}_{12}e^{i\underline{\mathbf{K}}_{1z}|z'|}\underline{\mathbf{K}}_{1z}^{-1}\underline{\mathbf{D}}_1^{-1}\mathbf{f}_1(\rho'), \tag{31}$$

$$\pm\rho A_{1\rho} = -\frac{I}{2p_1}\mathbf{f}_1^t(\rho)\underline{\mathbf{K}}_{1z}[\pm e^{i\underline{\mathbf{K}}_{1z}|z-z'|} - e^{i\underline{\mathbf{K}}_{1z}z}\underline{\mathbf{R}}_{12}e^{i\underline{\mathbf{K}}_{1z}|z'|}]\underline{\mathbf{K}}_{1z}^{-1}\underline{\mathbf{D}}_1^{-1}\mathbf{f}_1(\rho'), \tag{32}$$

and

$$\pm\rho A_{2\rho} = -\frac{I}{2p_2}\mathbf{f}_2^t(\rho)\underline{\mathbf{K}}_{2z}e^{i\underline{\mathbf{K}}_{2z}z}\underline{\mathbf{T}}_{12} \times e^{i\underline{\mathbf{K}}_{1z}|z'|}\underline{\mathbf{K}}_{1z}^{-1}\underline{\mathbf{D}}_1^{-1}\mathbf{f}_1(\rho'). \tag{33}$$

In the above, $\mathbf{f}_i$ are column vectors containing the eigenfunctions $f_{i\alpha}$, $\underline{\mathbf{K}}_{iz}$, and $\underline{\mathbf{D}}_i$ are diagonal matrices containing $k_{i\alpha z}$, and $\underline{\mathbf{D}}_\alpha$, $\underline{\mathbf{R}}_{12}$, and $\underline{\mathbf{K}}_{12}$ are reflection and transmission matrices for a wave incident from region I to region II.

Matching boundary conditions for the fields at $z = 0$, we have

$$\mathbf{f}_1^t(\rho)(\underline{\mathbf{I}} + \underline{\mathbf{R}}_{12}) = \mathbf{f}_2^t(\rho)\underline{\mathbf{T}}_{12}, \tag{34}$$

and

$$\mathbf{f}_1^t(\rho)\underline{\mathbf{K}}_{1z}(\underline{\mathbf{I}} - \underline{\mathbf{R}}_{12}) = \frac{p_1}{p_2}\mathbf{f}_2^t(\rho)\underline{\mathbf{K}}_{2z}\underline{\mathbf{T}}_{12}. \tag{35}$$

From our earlier discussion, $\mathbf{f}_1(\rho)$ and $\mathbf{f}_2(\rho)$ are related to the same basis set as

$$\mathbf{f}_1(\rho) = \underline{\mathbf{b}}_1\mathbf{g}(\rho),$$

and

$$\mathbf{f}_2(\rho) = \underline{\mathbf{b}}_2\mathbf{g}(\rho), \tag{36}$$

so

$$\mathbf{f}_2(\rho) = \underline{\mathbf{b}}_2\underline{\mathbf{b}}_1^{-1}\mathbf{f}_1(\rho). \tag{37}$$

In the above, $\underline{\mathbf{b}}_i$ is a matrix whose rows contain the eigenvectors, and $\mathbf{g}(\rho)$ is a column vector containing the functions $g_n(\rho)$. Hence equations (34) and (35) can be reexpressed as

$$\mathbf{g}^t(\rho)\underline{\mathbf{b}}_1^t(\underline{\mathbf{I}} + \underline{\mathbf{R}}_{12}) = \mathbf{g}^t(\rho)\underline{\mathbf{b}}_2^t\underline{\mathbf{T}}_{12}, \tag{38}$$

and

$$\mathbf{g}^t(\rho)\underline{\mathbf{b}}_1^t\underline{\mathbf{K}}_{1z}(\underline{\mathbf{I}} - \underline{\mathbf{R}}_{12}) = \frac{p_1}{p_2}\mathbf{g}^t(\rho)\underline{\mathbf{b}}_2^t\underline{\mathbf{K}}_{2z}\underline{\mathbf{T}}_{12}. \tag{39}$$

Since equation (38) is to be satisfied for all ρ, we conclude that

$$\underline{\mathbf{I}} + \underline{\mathbf{R}}_{12} = (\underline{\mathbf{b}}_2\underline{\mathbf{b}}_1^{-1})^t\underline{\mathbf{T}}_{12}. \tag{40}$$

For equation (39), we multiply by $(1/\rho p_1)\mathbf{g}(\rho)$ and integrate from a to ∞ to eliminate the ρ dependence. Since

$$\int_a^\infty d\frac{1}{\rho p_i}\mathbf{g}(\rho)\mathbf{g}^t(\rho) = \underline{\mathbf{G}}_i, \tag{41}$$

we have

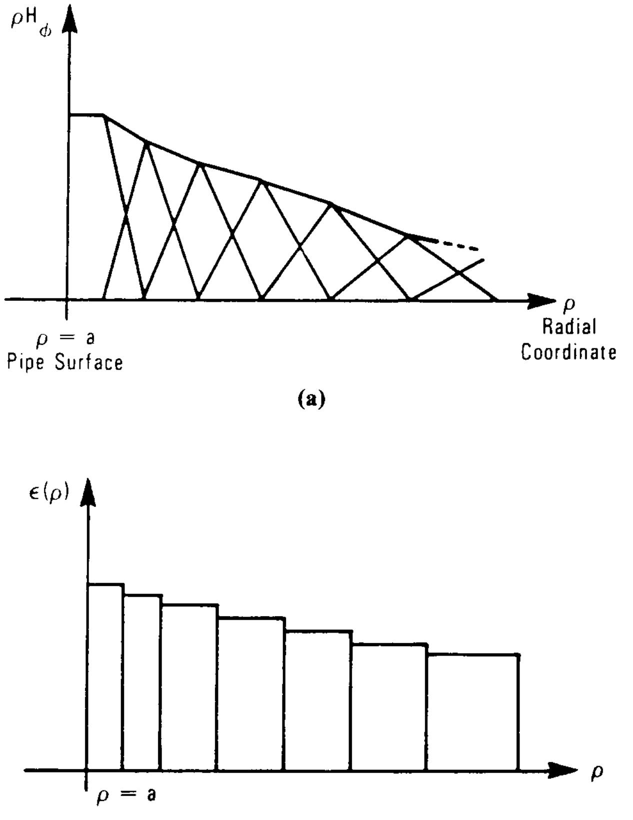

FIG. 2. The field is approximated with a piece-wise linear function while the radial electrical profile is assumed piece-wise constant.

$$\mathbf{G}_1\mathbf{b}_1^t\mathbf{K}_{1z}(\mathbf{I} - \mathbf{R}_{12}) = \mathbf{G}_2\mathbf{b}_2^t\mathbf{K}_{2z}\mathbf{T}_{12}. \tag{42}$$

We simplify the above since we know that

$$\mathbf{b}_i\mathbf{G}_i\mathbf{b}_i^t = \mathbf{D}_i \tag{43}$$

is a diagonal matrix. Hence equation (42) becomes

$$\mathbf{D}_1\mathbf{K}_{1z}(\mathbf{I} - \mathbf{R}_{12}) = \mathbf{b}_1\mathbf{b}_2^{-1}\mathbf{D}_2\mathbf{K}_{2z}\mathbf{T}_{12}. \tag{44}$$

Eliminating $\mathbf{T}_{12}$ from equations (40) and (44), we show that

$$\mathbf{D}_1\mathbf{K}_{1z}(\mathbf{I} - \mathbf{R}_{12}) = \mathbf{b}_1\mathbf{b}_2^{-1}\mathbf{D}_2\mathbf{K}_{2z}(\mathbf{b}_1\mathbf{b}_2^{-1})^t(\mathbf{I} + \mathbf{R}_{12}). \tag{45}$$

Solving the above, we obtain

$$\begin{aligned}\mathbf{R}_{12} = {} & [\mathbf{D}_1\mathbf{K}_{1z} + \mathbf{b}_1\mathbf{b}_2^{-1}\mathbf{D}_2\mathbf{K}_{2z}(\mathbf{b}_1\mathbf{b}_2^{-1})^t]^{-1} \\ & \times [\mathbf{D}_1\mathbf{K}_{1z} - \mathbf{b}_1\mathbf{b}_2^{-1}\mathbf{D}_2\mathbf{K}_{2z}(\mathbf{b}_1\mathbf{b}_2^{-1})^t],\end{aligned} \tag{46}$$

and

$$\begin{aligned}\mathbf{T}_{12} &= (\mathbf{b}_1\mathbf{b}_2^{-1})^t(\mathbf{I} + \mathbf{R}_{12}) \\ &= 2(\mathbf{b}_1\mathbf{b}_2^{-1})^t \\ &\quad \times [\mathbf{D}_1\mathbf{K}_{1z} + \mathbf{b}_1\mathbf{b}_2^{-1}\mathbf{D}_2\mathbf{K}_{2z}(\mathbf{b}_1\mathbf{b}_2^{-1})^t]^{-1}\mathbf{D}_1\mathbf{K}_{1z}.\end{aligned} \tag{47}$$

Note that these reflection and transmission operators obtained also can be derived by solving the surface integral equation associated with the bed boundary. However, we obtained them through indirect means, that is, by first finding the matrix representation of the differential operator in the original differential equation. The matrix representation of the Green's function is the inverse of the matrix representation of the differential operator. Our procedure is valid for boreholes of arbitrary inhomogeneous electrical profiles having many bed boundaries and radial inhomogeneous layers. However, the programming task becomes more tedious as the number of bed boundaries increases.

In the surface integral equation approach, the Green's function of the geometry of each bed has to be derived first. The matrix representation of the Green function, which is an integral operator, is then derived by using, for example, the method of moment or Galerkin's method (Strang and Fix, 1973; Harrington, 1968). The derivation of the matrix representation of the Green's integral operator usually requires laborious numerical integration.

RESULTS AND DISCUSSION

In this section, we discuss the results of the calculation using the theory we have outlined. We chose two types of expansion functions for $g_n(\rho)$ in equation (9): one type where $g_n(\rho)$ is a Hermitian-Gaussian function, and another where it is a triangular function as shown in Figure 2a.

The Hermitian-Gaussian functions have the advantage that a few of them may approximate the radial field profile described by $f_\alpha(\rho)$ in equation (5). The disadvantage is that the integrations in equations (13b) and (14) have to be performed numerically. The use of a triangular function is similar to the piece-wise linear approximation of the field, and for our calculation, a minimum of about 45 basis functions is needed to approximate the field radially. However, if we make approximations on the inhomogeneity profile around the borehole, such as assuming that they are piece-wise constant functions as shown in Figure 2b, we may perform the integrations in equations (13) and (14) analytically, thus enabling the matrix elements to be computed efficiently.

In theory, we have to approximate the field with triangular functions over an infinite support, i.e., from the center of the borehole or the center pipe surface $\rho = a$ to $\rho = \infty$. However, in the case of lossy media, the field is exponentially small a few skin depths from the borehole. Therefore, we only need to approximate the field over a finite support, i.e., from the borehole center to ρ_{max} where ρ_{max} is a few skin depths. Also, since the measurement of the field is done in the borehole, we are required to approximate the field more accurately close to the borehole. This can be achieved by choosing a small step size for the triangular function inside and close to the borehole, while a larger step size is used far from the borehole, as indicated in Figure 2.

Figure 3 shows the calculation of the TM wave response using our approach. This calculation supposes a 3.5 inch radius drill stem of infinite length with one transmitter and two receivers mounted on it. We plot the relative phases and amplitudes at the receivers as the tool moves across the bed boundary at $z = 0$. The calculation was performed using three and nine Hermitian-Gaussian polynomials, and with 45 triangular functions. Since the geometry excludes the borehole, the calculation compares with an exact solution obtained with the Fourier integral technique. We note that the nine-polynomial

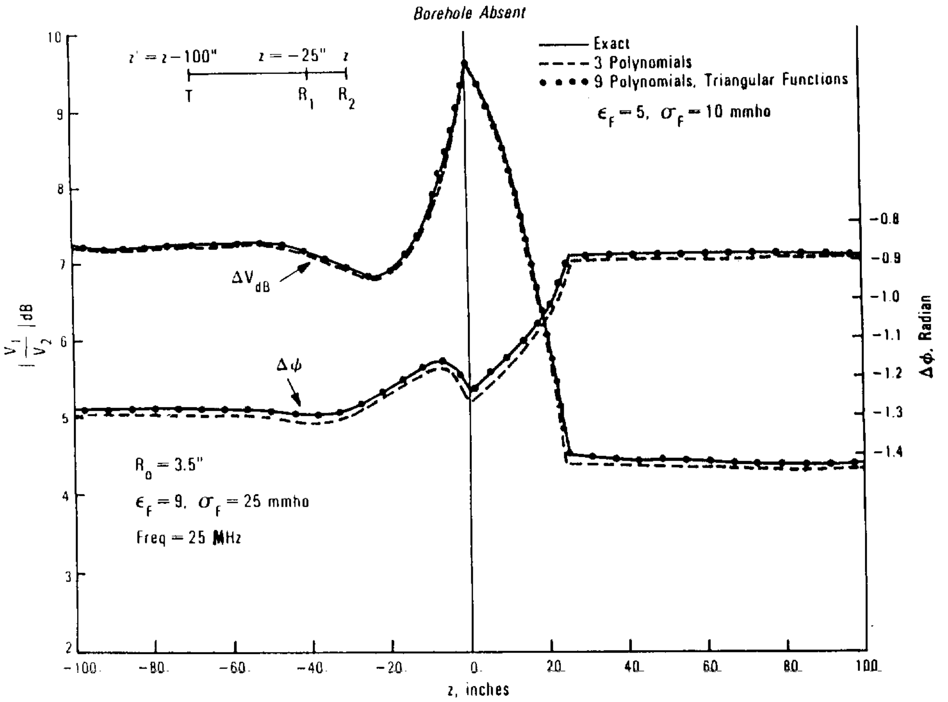

FIG. 3. Comparison of our approach using Hermitian-Gaussian polynomial or triangular basis functions and the exact Fourier integral technique for TM polarization.

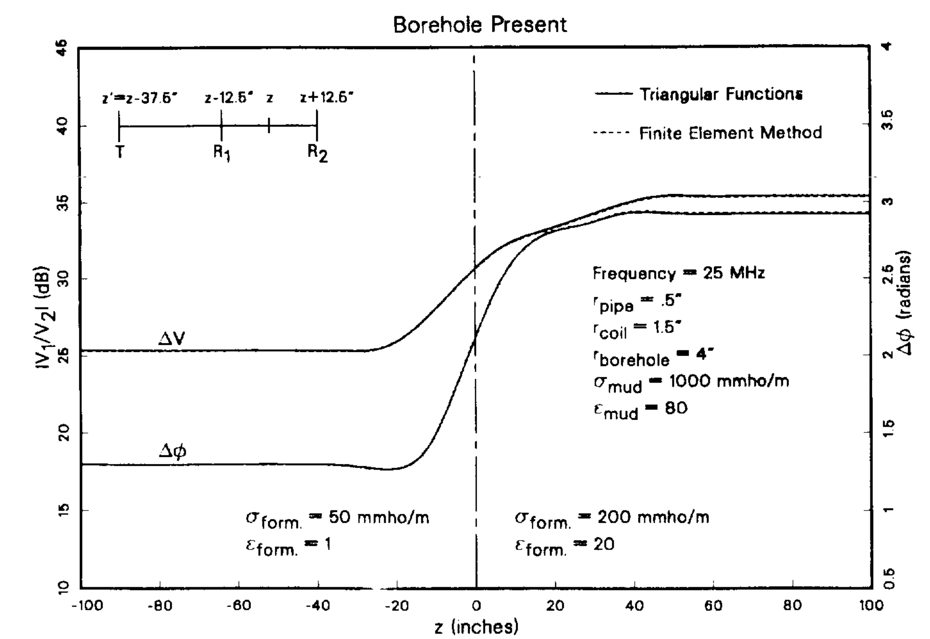

FIG. 4. Comparison of our approach using triangular basis functions and the finite-element method for TE polarization at dielectric logging frequency of 25 MHz. Our method is about 250 times faster than the FEM.

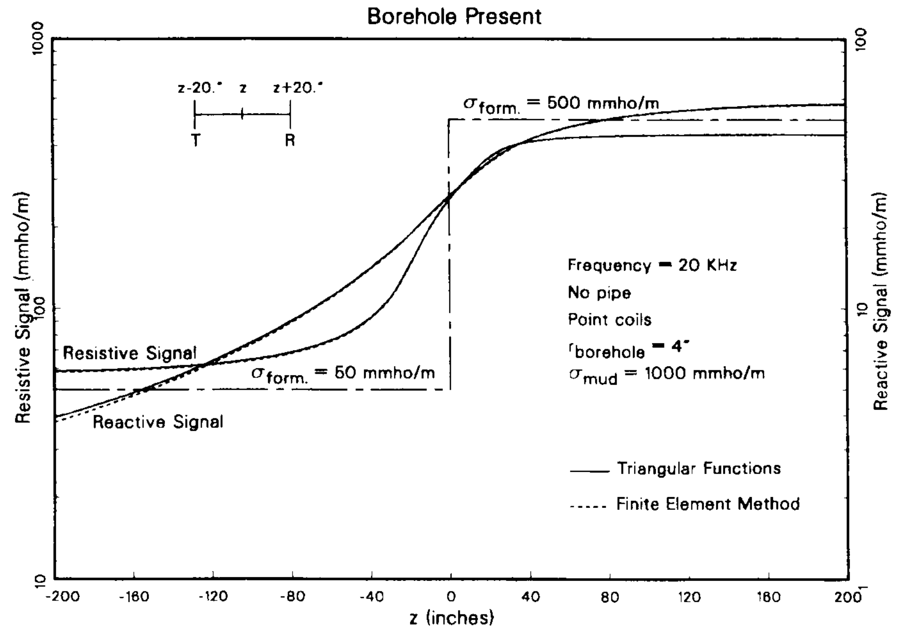

FIG. 5. Comparison with FEM at 20 kHz for a two-coil induction sonde for the R- and X-signal.

Hermitian-Gaussian function and the triangular function cases are in excellent agreement with the exact solution. When three polynomials are used, the computation is very efficient and can serve as a quick estimate of the response of the tool in the presence of the bed boundary. Note that there are cusps in the response curves corresponding to when the receivers cross the bed boundary.

Figure 4 shows the calculation of the TE wave response. This calculation is relevant to the dielectric logging tools at 25 MHz. The response shows the relative phases and amplitudes of the

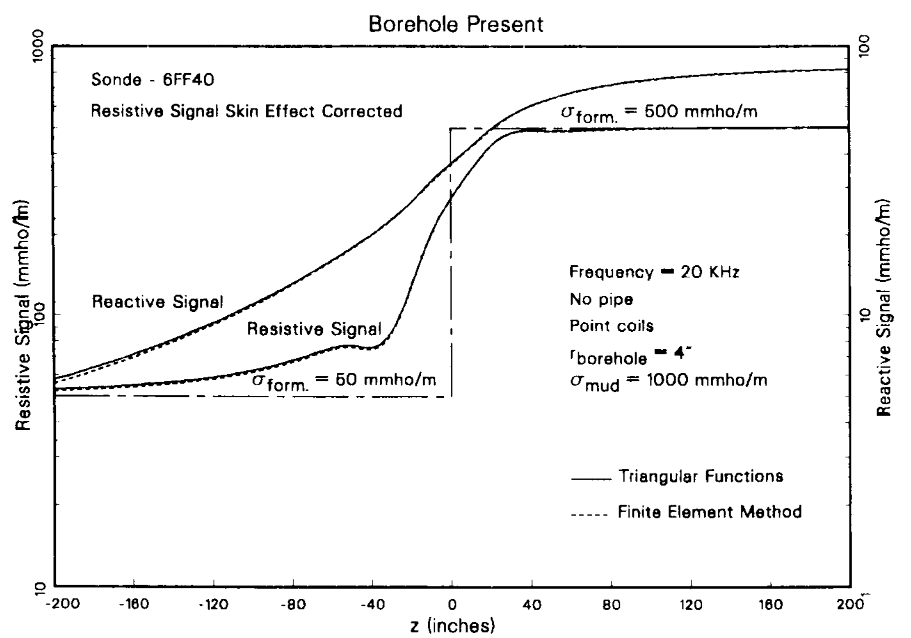

FIG. 6. Comparison with FEM for a 6FF40 induction sonde with skin effect correction.

signals at the two receivers as the tool moves across a bed boundary. The comparison with the finite-element method (Anderson and Chang, 1982; Anderson and Chang, 1983) is excellent and it is about 250 times faster. Note the absence of cusps in the TE response compared to the TM response.

Figures 5 and 6 show the applications of the method to induction logging. In Figure 5, a two-coil sonde is moving across a bed boundary within a borehole. The R- and X-signals are computed and the agreement with the finite-element method at 20 kHz is excellent, while our method is about 250 times faster. Figure 6 illustrates an application to the 6FF40 induction sonde with skin effect correction. The skin effect correction is chosen such that the sonde reads the correct conductivity for a homogeneous formation of 0.50 S.

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APPENDIX
RECIPROCITY AND ENERGY CONSERVATION

If we have a current loop source at (ρ_1, z_1), the field it generates satisfies the equation

$$D\rho A_\phi^{(1)} = -i\omega p I \rho \delta(\rho - \rho_1)\delta(z - z_1), \tag{A-1}$$

where D is the linear differential operator on the left-hand side of equation (3). Similarly, a current loop source at (ρ_2, z_2) generates a field that satisfies

$$D\rho A_\phi^{(2)} = -i\omega p I \rho \delta(\rho - \rho_2)\delta(z - z_2). \tag{A-2}$$

We can multiply equation (A-1) by $(1/p)A_\phi^{(2)}$ and integrate over space to obtain

$$\langle \rho A_\phi^{(2)}, D\rho A_\phi^{(1)} \rangle = -i\omega I \rho_1 A_\phi^{(2)}(\rho_1, z_1). \tag{A-3a}$$

Similarly, multiplying equation (49) by $(1/p)A_\phi^{(1)}$ and integrating, we have

$$\langle \rho A_\phi^{(1)}, D_\rho A_\phi^{(2)} \rangle = -i\omega I \rho_2 A_\phi^{(1)}(\rho_2, z_2). \tag{A-3b}$$

The inner products on the left-hand side of equations (A-3a) and (A-3b) are defined as

$$\langle f, g \rangle = \int_a^\infty \frac{d\rho}{\rho p} \int_{-\infty}^\infty dz\, fg. \tag{A-4}$$

Under this definition, the D operator is self-adjoint (Harrington, 1968),

$$\langle \rho A_\phi^{(2)}, D\rho A_\phi^{(1)} \rangle = \langle \rho A_\phi^{(1)}, D\rho A_\phi^{(2)} \rangle. \tag{A-5}$$

Consequently, we have

$$\rho_1 A_\phi^{(2)}(\rho_1, z_1) = \rho_2 A_\phi^{(1)}(\rho_2, z_2). \tag{A-6}$$

The above equation is a statement of the reciprocity theorem, that is, *the field at* (ρ_1, z_1) *due to a current loop source at* (ρ_2, z_2) *is the same as the field at* (ρ_2, z_2) *due to a current loop source at* (ρ_1, z_1).

The satisfaction of the reciprocity theorem is not an absolute check for the correctness of a solution, but it acts as a consistency check for possible programming errors. In fact, we show that our approximate solution satisfies reciprocity exactly. In order for equation (30) to satisfy reciprocity, we require that

$$\rho A_{1\phi}(\rho, z; \rho', z') = \rho' A_{1\phi}(\rho', z'; \rho, z). \tag{A-7}$$

From equation (30), we note that this is true only if

$$[\underline{\mathbf{R}}_{12}\underline{\mathbf{K}}_{1z}^{-1}\underline{\mathbf{D}}_1^{-1}]^t = \underline{\mathbf{D}}_1^{-1}\underline{\mathbf{K}}_{1z}^{-1}\underline{\mathbf{R}}_{12}^t = \underline{\mathbf{R}}_{12}\underline{\mathbf{K}}_{1z}^{-1}\underline{\mathbf{D}}_1^{-1}. \tag{A-8}$$

The above can be proven easily from equation (46).

In order for equation (31) to satisfy reciprocity, we require that equation (31) be the same as the case when a current loop source at (ρ, z) in region II generates a field at (ρ', z') in region I. In equation form, it is

$$-\frac{\omega I}{2}\mathbf{f}_2^t(\rho)e^{i\mathbf{K}_{2z}z}\underline{\mathbf{T}}_{12}e^{i\mathbf{K}_{1z}|z'|}\underline{\mathbf{K}}_{1z}^{-1}\underline{\mathbf{D}}_1^{-1}\mathbf{f}_1(\rho')$$

$$= -\frac{\omega I}{2}\mathbf{f}_1^t(\rho')e^{i\mathbf{K}_{1z}z'}\underline{\mathbf{T}}_{21}e^{i\mathbf{K}_{2z}|z|}\underline{\mathbf{K}}_{2z}^{-1}\underline{\mathbf{D}}_2^{-1}\mathbf{f}_2(\rho). \tag{A-9}$$

In order for the above equation to be satisfied, we require that

$$\underline{\mathbf{T}}_{12}\underline{\mathbf{K}}_{1z}^{-1}\underline{\mathbf{D}}_1^{-1} = \underline{\mathbf{K}}_{2z}^{-1}\underline{\mathbf{D}}_2^{-1}\underline{\mathbf{T}}_{21}^t. \tag{A-10}$$

The above can be proven from equation (47). $\underline{\mathbf{T}}_{21}$ is obtained from equation (47) by exchanging the subscripts 1 and 2.

Furthermore, in order for equation (31) to be itself when we take its transpose, we require

$$\underline{\mathbf{T}}_{12}\underline{\mathbf{K}}_{1z}^{-1}\underline{\mathbf{D}}_1^{-1}\underline{\mathbf{b}}_1\underline{\mathbf{b}}_2^{-1}$$

to be symmetrical. Also, we show easily that

$$\underline{\mathbf{b}}_1^t\underline{\mathbf{R}}_{12}(\underline{\mathbf{b}}_1^t)^{-1} = \underline{\mathbf{b}}_2^t\underline{\mathbf{R}}_{21}(\underline{\mathbf{b}}_2^t)^{-1}. \tag{A-11}$$

To prove global energy conservation at the boundary $z = 0$, we need to show that

$$\int_a^\infty d\rho\, \rho A_{1\phi}A_{1\rho}^* = \int_a^\infty d\rho\, A_{2\phi}A_{2\rho}^*\Big|_{z=0}. \tag{A-12}$$

From equations (30)–(33), this is the same as proving that

$$\mathbf{f}_1^t(\rho')\underline{\mathbf{K}}_{1z}^{-1}\underline{\mathbf{D}}_1^{-1}[\underline{\mathbf{I}} + \underline{\mathbf{R}}_{12}^t]\int_a^\infty d\rho\,\frac{1}{\rho p_1^*}\mathbf{f}_1(\rho)\mathbf{f}_1^{*t}(\rho)$$

$$\times\,\underline{\mathbf{K}}_{1z}^*[\underline{\mathbf{I}} - \underline{\mathbf{R}}_{12}^{t*}]\underline{\mathbf{K}}_{1z}^{*-1}\underline{\mathbf{D}}_1^*\mathbf{f}_1^*(\rho')$$

$$= \mathbf{f}_1^t(\rho')\underline{\mathbf{K}}_{1z}^{-1}\underline{\mathbf{D}}_1^{-1}\underline{\mathbf{T}}_{12}^t\int_a^\infty d\rho\,\frac{1}{\rho p_2^*}\mathbf{f}_2(\rho)\mathbf{f}_2^{*t}(\rho)$$

$$\times\,\underline{\mathbf{K}}_{2z}^*\underline{\mathbf{T}}_{12}^*\underline{\mathbf{K}}_{1z}^{*-1}\underline{\mathbf{D}}_1^{*-1}\mathbf{f}_1^*(\rho'). \tag{A-13}$$

We can show easily that

$$\int_a^\infty d\rho\,\frac{1}{\rho p_i^*}\mathbf{f}_i(\rho)\mathbf{f}_i^{*t}(\rho) = \underline{\mathbf{b}}_i\underline{\mathbf{G}}_i^*\underline{\mathbf{b}}_i^{t*}. \tag{A-14}$$

Hence, equation (A-13) is equivalent to

$$[\underline{\mathbf{I}} + \underline{\mathbf{R}}_{12}^t]\underline{\mathbf{b}}_1\underline{\mathbf{G}}_1^*\underline{\mathbf{b}}_1^{t*}\underline{\mathbf{K}}_{1z}^*[\underline{\mathbf{I}} - \underline{\mathbf{R}}_{12}^*]$$

$$= \underline{\mathbf{T}}_{12}^t\underline{\mathbf{b}}_2\underline{\mathbf{G}}_2^*\underline{\mathbf{b}}_2^*\underline{\mathbf{K}}_{2z}^*\underline{\mathbf{T}}_{12}^*. \tag{A-15}$$

The equality of equation (A-15) follows from Equations (40) and (42).

Our approximate solution satisfies reciprocity exactly, because we have maintained the self-adjointness of our differential operator even when we have replaced it with its matrix representation. Also, when we match boundary condition at the bed boundary to obtain the reflection and transmission matrices, it is equivalent to solving a surface integral equation at the boundary by Galerkin's variational method, which maintains reciprocity and conserves energy. However, by working with the differential equation from the very beginning, we have bypassed the need for a Green's operator in functional form. The formulation of a surface integral equation requires that the Green's function be expressible in terms of Bessel functions which are difficult to evaluate.

VARIATIONAL PROPERTIES

The eigenvalue problem that we wish to solve for equation (6) can be written in operational form as

$$L_1 f = k_z^2 f, \tag{A-16}$$

where

$$L_1 = \rho p\frac{\partial}{\partial\rho}\frac{1}{\rho p}\frac{\partial}{\partial\rho} + \omega^2\mu\varepsilon. \tag{A-17}$$

We write a variational expression for k_z^2, the eigenvalue, as

$$k_z^2 = \frac{\langle u, L, \mu\rangle}{\langle u, u\rangle}, \tag{A-18}$$

where L_1 is self-adjoint with the definition of inner product given in equation (12), i.e., $\langle g, L_1 f\rangle = \langle f, L_1 g\rangle$. We can show that when f satisfies equation (A-16), it is at the stationary point of equation (A-18). For example, if we have a trial function $u = f + \varepsilon v$, we find that

$$\tilde{k}_z^2 = \frac{\langle f, L_1 f\rangle}{\langle f, f\rangle}\left[1 + 2\varepsilon\left(\frac{\langle v, L_1 f\rangle}{\langle f, L_1 f\rangle} - \frac{\langle v, f\rangle}{\langle f, f\rangle}\right) + \cdots\right]. \tag{A-19}$$

Since $L_1 f = k_z^2 f$, the first variation vanishes, implying its stationarity.

The Rayleigh-Ritz procedure in solving equation (A-18) assumes that

$$u = \sum_{a=1}^N a_n g_n, \tag{A-20}$$

and requires that

$$\frac{\partial k_z^2}{\partial a_n} = 0, \qquad n = 1, \cdots, N. \tag{A-21}$$

We find that in order for equation (A-21) to be true, we require that

$$\sum_{n=1}^N a_n\langle g_n, L_1 g_m\rangle = k_z^2\sum_{n=1}^N a_n\langle g_n, g_m\rangle. \tag{A-22}$$

The above is the same as equation (13).

When we find the solution to equation (3) with the procedure of equations (18)–(24), it is equivalent to the use of Galerkin's variational method. Given the differential equation

$$L_2 f = g, \tag{A-23}$$

where in our example

$$L_2 = \left[\rho p \frac{\partial}{\partial \rho} \frac{1}{\rho p} \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} + k^2(\rho)\right], \tag{A-24a}$$

$$f = \rho A_\phi, \tag{A-24b}$$

and

$$g = -i\omega p I \rho' \delta(\rho - \rho')\delta(z - z'). \tag{A-24c}$$

We can write a variational expression for a quantity for which we are interested in

$$\langle f, h \rangle = \frac{\langle f, h \rangle \langle f^a, g \rangle}{\langle L_2 f, f^a \rangle}, \tag{A-25}$$

where

$$L_2 f^a = h, \tag{A-26}$$

corresponding to an adjoint problem. We assume that L_2 is self-adjoint. If we take the first variation of f and f^a in equation (A-25), i.e., $u = f + \varepsilon v$, $u^a = f^a + \varepsilon v^a$, we can show that it vanishes. The first variation is

$$\langle f, h \rangle = \frac{\langle f, h \rangle \langle f^a, g \rangle}{\langle L_2 f, f^a \rangle} \left[1 + \varepsilon \frac{\langle v, h \rangle}{\langle f, h \rangle} + \varepsilon \frac{\langle v^a, g \rangle}{\langle f^a, g \rangle} - \varepsilon \frac{\langle L_2 v, f^a \rangle}{\langle L_2 f, f^a \rangle} - \varepsilon \frac{\langle L_2 f, v^a \rangle}{\langle L_2 f, f^a \rangle} + \cdots \right], \tag{A-27}$$

which by equations (A-23), (A-26), and the self-adjointness of L_2 vanish. If we apply the Rayleigh-Ritz procedure to equation (A-25) to obtain a variational result, we show that it is equivalent to solving for the solutions of equations (A-23) and (A-26) using Galerkin's method. It can also be shown that the finite-element method is a consequence of applying Galerkin's method to a differential equation.