# Response of a Source on Top of a Vertically Stratified Half-Space

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***Abstract*—The solution of the response of a source on top of a horizontally stratified half-space is well-known. However, when the half-space is vertically stratified, the problem can only be solved with numerical methods like the finite element method. Here a semi-analytic approach to solve such a problem is presented. The three-dimensional variation of the problem is reduced to two-dimensional variation by Fourier transform in one coordinate variable. The remaining two-dimensional problem is solved by finding the eigensolution in each of the half-spaces. The eigensolutions of each region are found from the partial differential equation directly using the same basis set of expansion functions. This makes the calculation of the reflection and transmission operators very efficient. The reflection and transmission operators account for the mode-conversion, reflection, and transmission of the waves. With the reflection and transmission operators, the field everywhere can be calculated. The solution reduces to that of the Sommerfeld's half-space problem when the two half-spaces are homogeneous.**

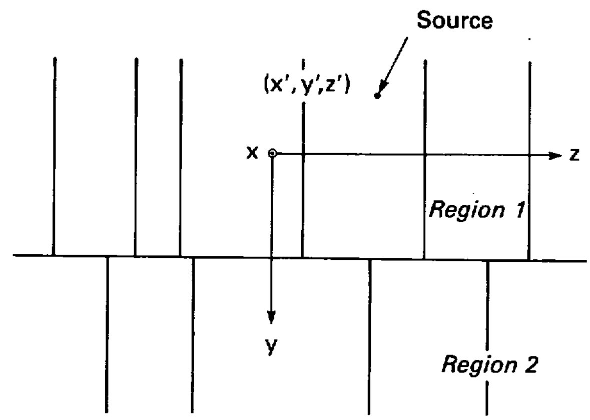


Fig. 1. Radiation of a source in a geometry that consists of two half-spaces that have vertical stratification.

## I. INTRODUCTION

THE RADIATION of a point or a line source on top of a vertically stratified half-space is an unsolved problem except by the finite element method. Due to the inherent large size of the scatterer, the implementation of the finite element problem requires large storage and also large amounts of computation [1]. We shall describe in this paper a way to solve this problem without using an inordinately large amount of storage by using a semi-analytic method [2]-[4].

The radiation of a source on top of a half-space shown in Fig. 1 resembles the scattering of waves by discontinuities. This is an important phenomenon that occurs in geophysical prospecting, optics, and in many other areas of wave propagation. The problem can be solved by first fomulating an integral equation relating the field excited in the respective half-spaces. The solution of such integral equation can be difficult to obtain, and it has been treated with the variational approach [5]-[11], Wiener-Hopf method [12]-[15], Neumann series expansion [16]-[17] and mode matching [18]-[23]. A more expedient method to solve this problem is to solve the differential equations in the respective half-space to obtain the eigensolutions with the same basis set. Once the eigensolutions are obtained under the same basis, the reflection and transmission operators are easily obtained by requiring the continuity of the tangential electric and magnetic fields across the discontinuity. Such method gives rise to a solution that is variational, and satisfies reciprocity and energy conservation as shown in [2]. The method is also efficient on the computer compared to the finite element method [2], [3].

## II. FORMULATION

Before we consider the problem of a source radiating on top of a vertically stratified half-space, let us consider a geometry of

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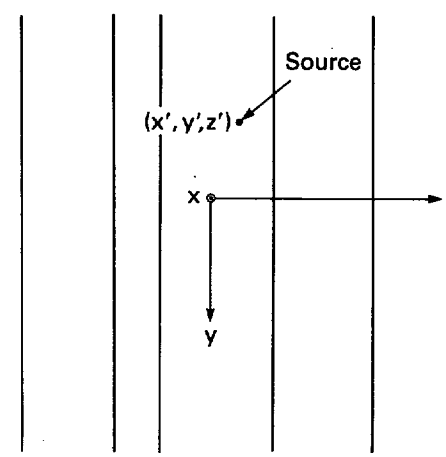


Fig. 2. Radiation of the source in a geometry without the discontinuity in Fig. 1 and solvable by the Fourier transform technique.

whole-space of vertical stratification as shown in Fig. 2. This problem can be solved by the Fourier integral technique. However, we shall discuss a more general method of solving this problem. This general method will make the solution of the vertically stratified half-space problem tenable.

Assuming only current sources, and that $\epsilon(z)$ is an arbitrary function of $z$, we can show from Maxwell's equation that the fields satisfy the following equations:

$$\mu\nabla \times \mu^{-1}\nabla \times \bar{E} - k^2\bar{E} = i\omega\mu\bar{J}, \tag{1}$$

$$\epsilon\nabla \times \epsilon^{-1}\nabla \times \bar{H} - k^2\bar{H} = \epsilon\nabla \times \epsilon^{-1}\bar{J}. \tag{2}$$

The time dependence $\exp(-i\omega t)$ is suppressed. Taking only the $z$-components of the above equations, we easily deduce that

$$\left(\nabla_s^2 + \frac{\partial}{\partial z}\,\epsilon^{-1}\,\frac{\partial}{\partial z}\,\epsilon + k^2\right) E_z = -i\omega\mu J_z + \frac{\partial}{\partial z}\,\frac{\rho}{\epsilon} \tag{3}$$

$$\left(\nabla_s^2 + \frac{\partial}{\partial z}\,\mu^{-1}\,\frac{\partial}{\partial z}\,\mu + k^2\right) H_z = -(\nabla_s \times \bar{J}_s)_z \tag{4}$$

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where $\nabla_s^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ and $\rho$ is the electric charge density. The first equation in the preceding corresponds to transverse magnetic (TM) waves while the second equation corresponds to transverse electric (TE) waves with respect to the $z$-direction. To reduce the dimension of the equations, we Fourier transform in the $x$-variable, and obtain

$$\left[\frac{\partial^2}{\partial y^2} + \epsilon \frac{\partial}{\partial z} \epsilon^{-1} \frac{\partial}{\partial z} + k^2 - k_x^2\right] \epsilon\tilde{E}_z = -i\omega\mu\epsilon\tilde{J}_z + \epsilon \frac{\partial}{\partial z} \tilde{\rho}/\epsilon \tag{5}$$

$$\left[\frac{\partial^2}{\partial y^2} + \mu \frac{\partial}{\partial z} \mu^{-1} \frac{\partial}{\partial z} + k^2 - k_x^2\right] \mu\tilde{H}_z = -\mu(\nabla_s \times \tilde{J}_s)_z \tag{6}$$

where the tilde denotes a transformed quantity and $k_x$ is the Fourier transform variable in the $x$-direction. To solve the above equations, we first find the eigensolutions of the equations in the absence of the source. The eigensolutions assume the general forms due to the $y$-independence of the electrical property of the medium.

$$f_\alpha(z) e^{ik_{\alpha y} y}. \tag{7}$$

As such

$$[L_p - k_{p\alpha\rho}^2] f_{p\alpha} = 0 \tag{8}$$

where $p = \epsilon$ for TM field, and $p = \mu$ for TE field. Also, $k_{k\alpha p}^2 = k_x^2 + k_{p\alpha y}^2$, and $L_p = p(\partial/\partial z)p^{-1}\partial/\partial z + k^2$. To solve the above we let

$$f_{p\alpha} = \sum_i b_{p\alpha i} S_i(z). \tag{9}$$

Substituting the above into (8), we obtain

$$\sum_i b_{p\alpha i} [L_p - k_{p\alpha\rho}^2] S_i(z) = 0. \tag{10}$$

Multiplying the above by $S_j(z)/p$ and integrating, we obtain

$$\sum_i b_{p\alpha i} [\langle S_j, p^{-1} L_p S_i\rangle - k_{p\alpha\rho}^2 \langle S_j, p^{-1} S_i\rangle] = 0 \tag{11}$$

where the inner product is $\langle f, g\rangle = \int_{-\infty}^{\infty} dz f(z) g(z)$. Alternatively, the above can be written as

$$[\bar{\bar{L}}_p - k_{p\alpha\rho}^2 \bar{\bar{p}}^{-1}] \cdot \bar{b}_{p\alpha} = 0 \tag{12}$$

where

$$[\bar{\bar{L}}_p]_{ij} = \langle S_i, p^{-1} L_p S_j\rangle, \quad [\bar{\bar{p}}^{-1}]_{ij} = \langle S_i, p^{-1} S_j\rangle. \tag{13}$$

The above matrices are symmetrical as a consequence of reciprocity.

We can solve (12) for the eigenvalues $k_{p\alpha\rho}^2$ and the eigenvectors $\bar{b}_{p\alpha}$. Knowing $\bar{b}_{p\alpha}$, we can find the eigenfunctions from (9). We can further show that

$$\langle f_{p\alpha}, p^{-1} f_{p\beta}\rangle = \bar{b}_{p\alpha}^t \cdot \bar{\bar{p}}^{-1} \cdot \bar{b}_{p\alpha} = \delta_{\alpha\beta}, \tag{14}$$

where we have normalized the eigenvectors so that the eigenfunctions are orthonormal, in the above inner product.

With the eigenfunctions available, we can proceed to solve (5) and (6) with the eigenfunction expansion method. The method is general, but for the case when the current source is a Hertzian electric dipole polarized in the $y$-direction, the solutions are

$$\epsilon\tilde{E}_z = \mp \frac{Il}{2i\omega\epsilon(z')} \sum_\alpha e^{ik_{\epsilon\alpha y}|y-y'|} f_{\epsilon\alpha}(z) f'_{\epsilon\alpha}(z) f'_{\epsilon\alpha}(z') \tag{15}$$

$$\mu\tilde{H}_z = -\frac{Ilk_x}{2} \sum_\alpha e^{ik_{\mu\alpha y}|y-y'|} k_{\mu\alpha y}^{-1} f_{\mu\alpha}(z) f_{\mu\alpha}(z') \tag{16}$$

where $Il$ denotes the strength of the Hertzian dipole at $(x', y', z')$.

For clarity and simplicity, the preceding can be written in matrix notation as

$$\epsilon\tilde{E}_z = \mp \frac{Il}{2i\omega\epsilon(z')} \bar{f}_\epsilon^{\prime t}(z) \cdot e^{i\bar{\bar{K}}_{\epsilon y}|y-y'|} \cdot \bar{f}'_\epsilon(z'), \tag{17}$$

$$\mu\tilde{H}_z = -\frac{Ilk_x}{2} \bar{f}_\mu^t(z) \cdot e^{i\bar{\bar{K}}_{\mu y}|y-y'|} \cdot \bar{\bar{K}}_{\mu y}^{-1} \cdot \bar{f}_\mu(z'), \tag{18}$$

where $\bar{f}_p$, and $\bar{f}'_p$ are column vectors containing $f_{p\alpha}$, and $f'_{p\alpha}$, respectively, and $\bar{\bar{K}}_{py}$ are diagonal matrices containing $k_{p\alpha y}$, where $k_{p\alpha y}$ is derived from the dispersion relation after (8). From matrix theory, it is seen that $\exp[i\bar{\bar{K}}_{py} y]$ is diagonal matrix with $[\exp(i\bar{\bar{K}}_{py} y)]_{\alpha\alpha} = \exp(ik_{p\alpha y} y)$.

In the presence of a discontinuity at $y = d$ as shown in Fig. 1, we require $H_x$, $H_z$, $E_x$, and $E_z$ to be continuous across that discontinuity. It can be shown from Maxwell's equations that the field components transverse to $z$ (denoted by $s$) are

$$\bar{E}_s = L_\mu^{-1}\left[\mu \frac{\partial}{\partial z} \mu^{-1} \nabla_s E_z + i\omega\mu \nabla_s \times \bar{H}_z\right], \tag{19}$$

$$\bar{H}_s = L_\epsilon^{-1}\left[\epsilon \frac{\partial}{\partial z} \epsilon^{-1} \nabla_s H_z - i\omega\epsilon \nabla_s \times \bar{E}_z\right], \tag{20}$$

where $L_p$ is the differential operator as in (8). Hence, $L_p^{-1}$ is an integral operator. Since the horizontal discontinuity couples the TE and TM fields, we introduce a new way to write (17) and (18):

$$\bar{A}_z = \bar{\bar{E}}^{-1} \cdot \bar{\bar{F}}^t(z) \cdot \bar{\bar{U}} \cdot e^{i\bar{\bar{K}}|y-y'|} \cdot \bar{F}_s(z'), \tag{21}$$

where

$$\bar{A}_z = \begin{bmatrix} \tilde{E}_z \\ \tilde{H}_z \end{bmatrix}, \quad \bar{\bar{E}} = \begin{bmatrix} \epsilon & 0 \\ 0 & \mu \end{bmatrix}, \quad \bar{\bar{F}}^t(z) = \begin{bmatrix} \bar{f}_\epsilon^{\prime t}(z) & 0 \\ 0 & \bar{f}_\mu^t(z) \end{bmatrix},$$

$$\bar{\bar{U}} = \begin{bmatrix} \operatorname{sgn}(y-y') & 0 \\ 0 & 1 \end{bmatrix}, \quad \bar{\bar{K}} = \begin{bmatrix} \bar{\bar{K}}_{\epsilon y} & 0 \\ 0 & \bar{\bar{K}}_{\mu y} \end{bmatrix},$$

$$\bar{F}_s(z') = \begin{bmatrix} -\dfrac{Il}{2i\omega\epsilon(z')} \bar{f}'_\epsilon(z') \\ -\dfrac{Ilk_x}{2} \bar{\bar{K}}_{\mu y}^{-1} \cdot \bar{f}_\mu(z') \end{bmatrix}. \tag{22}$$

Similarly using (19) and (20) for the $x$-component, we have

$$\bar{A}_x = \begin{bmatrix} \tilde{H}_x \\ \tilde{E}_x \end{bmatrix} = \bar{\bar{\lambda}}^{-1} \cdot \bar{\bar{M}} \cdot \bar{A}_z \tag{23}$$

where

$$\bar{\bar{\lambda}} = \begin{bmatrix} L_\epsilon & 0 \\ 0 & L_\mu \end{bmatrix},$$

$$\bar{\bar{M}} = \begin{bmatrix} -i\omega\epsilon \dfrac{\partial}{\partial y} & ik_x\epsilon \dfrac{\partial}{\partial z}\, \epsilon^{-1} \\ ik_x\mu \dfrac{\partial}{\partial z}\, \mu^{-1} & i\omega\mu \dfrac{\partial}{\partial y} \end{bmatrix}.$$

Alternatively, (23) can be rewritten as

$$\bar{A}_x = \bar{\bar{\lambda}}^{-1}(z) \cdot \bar{\bar{E}}(z) \cdot \bar{\bar{N}}_{\pm}(z) \cdot \bar{\bar{U}} \cdot e^{i\bar{\bar{K}}|y-y'|} \cdot \bar{F}_s(z'), \tag{25}$$

where

$$\bar{\bar{N}}_{\pm}(z) = \begin{bmatrix} \pm\omega\epsilon^{-1}\bar{f}_\epsilon^t \cdot \bar{\bar{K}}_{\epsilon y} & ik_x \dfrac{\partial}{\partial z}\, \epsilon^{-1}\mu^{-1}\bar{f}_\mu^t \\ ik_x \dfrac{\partial}{\partial z}\, \epsilon^{-1}\mu^{-1}\bar{f}_\epsilon^t & \mp\omega\mu^{-1}\bar{f}_\mu^t \cdot \bar{\bar{K}}_{\mu y} \end{bmatrix} \tag{26}$$

where the upper sign is chosen when $\operatorname{sgn}(y - y') > 0$ and vice versa.

$$\bar{\bar{\Omega}}_{i\pm} = \langle \bar{\bar{S}}, \bar{\bar{N}}_{i\pm} \cdot \bar{\bar{S}}^t \rangle = \begin{bmatrix} \pm\omega\bar{\bar{\epsilon}}_i^{-1} \cdot \bar{\bar{b}}^t_{\epsilon_i} \cdot \bar{\bar{K}}_{\epsilon_i y} & ik_x \left\langle \bar{S}, \dfrac{\partial}{\partial z}(\mu_i\epsilon_i)^{-1}\bar{S}^t \right\rangle \cdot \bar{\bar{b}}^t_{\mu_i} \\ ik_x \left\langle \bar{S}, \dfrac{\partial}{\partial z}\,(\mu_i\epsilon_i)^{-1}\bar{S}^t \right\rangle \cdot \bar{\bar{b}}^t_{\epsilon_i} & \mp\omega\bar{\bar{\mu}}_i^{-1} \cdot \bar{\bar{b}}^t_{\mu i} \cdot \bar{\bar{K}}_{\mu_i y} \end{bmatrix}. \tag{39}$$

In the presence of a discontinuity, as in Fig. 1, (21) is not a complete representation of $\bar{A}_z$. More appropriately, the field in region 1 contains a reflected component

$$\bar{A}_{1z} = \bar{\bar{E}}_1^{-1} \cdot \bar{\bar{F}}_1^t(z) \cdot [\bar{\bar{U}} \cdot e^{i\bar{\bar{K}}_1|y-y'|} + e^{-i\bar{\bar{K}}_1(y-d)} \cdot \bar{\bar{R}}_{12} \cdot e^{i\bar{\bar{K}}_1|d-y'|}] \cdot \bar{F}_{1s}(z'), \tag{27}$$

where the subscript one denotes the region, and $\bar{\bar{R}}_{12}$ is a reflection operator. The $x$-components can be derived via (25) and (26).

$$\bar{A}_{1x} = \bar{\bar{\lambda}}_1^{-1} \cdot \bar{\bar{E}}_1 \cdot [\bar{\bar{N}}_{1\pm} \cdot \bar{\bar{U}} \cdot e^{i\bar{\bar{K}}_1|y-y'|} + \bar{\bar{N}}_{1-} \cdot e^{-i\bar{\bar{K}}_1(y-d)} \cdot \bar{\bar{R}}_{12} \cdot e^{i\bar{\bar{K}}_1|d-y|}] \cdot \bar{F}_{1s}(z'). \tag{28}$$

In region 2,

$$\bar{A}_{2z} = \bar{\bar{E}}_2^{-1} \cdot \bar{\bar{F}}_2^t(z) \cdot e^{i\bar{\bar{K}}_2(y-d)} \cdot \bar{\bar{T}}_{12} \cdot e^{i\bar{\bar{K}}_1|d-y'|} \cdot \bar{F}_{1s}(z'), \tag{29}$$

$$\bar{A}_{2x} = \bar{\bar{\lambda}}_2^{-1} \cdot \bar{\bar{E}}_2 \cdot \bar{\bar{N}}_{2+} \cdot e^{i\bar{\bar{K}}_2(y-d)} \cdot \bar{\bar{T}}_{12} \cdot e^{i\bar{\bar{K}}_1|d-y'|} \cdot \bar{F}_{1s}(z') \tag{30}$$

where $\bar{\bar{T}}_{12}$ is a transmission operator. Matching boundary conditions at $y = d$, we have

$$\bar{\bar{E}}_1^{-1} \cdot \bar{\bar{F}}_1^t(z) \cdot (\bar{\bar{I}} + \bar{\bar{R}}_{12}) = \bar{\bar{E}}_2^{-1} \cdot \bar{\bar{F}}_2^t(z) \cdot \bar{\bar{T}}_{12}, \tag{31}$$

$$\bar{\bar{\lambda}}_1^{-1} \cdot \bar{\bar{E}}_1 \cdot (\bar{\bar{N}}_{1+} + \bar{\bar{N}}_{1-} \cdot \bar{\bar{R}}_{12}) = \bar{\bar{\lambda}}_2^{-1} \cdot \bar{\bar{E}}_2 \cdot \bar{\bar{N}}_{2+} \cdot \bar{\bar{T}}_{12}. \tag{32}$$

Defining

$$\bar{\bar{S}} = \begin{bmatrix} \bar{S} & 0 \\ 0 & \bar{S} \end{bmatrix}, \quad \bar{\bar{B}} = \begin{bmatrix} \bar{\bar{b}}_\epsilon & 0 \\ 0 & \bar{\bar{b}}_\mu \end{bmatrix} \tag{33}$$

where $[S]_i = S_i(z)$ and $[\bar{\bar{b}}_p]_{ij} = b_{pij}$ in (9). Letting $\bar{\bar{F}}_i^t = \bar{\bar{S}}^t \cdot \bar{\bar{B}}_i^t$, and multiplying (31) and (32) by $S$ and integrating, we have

$$\bar{\bar{\alpha}}_1^{-1} \cdot \bar{\bar{B}}_1^t \cdot (\bar{\bar{I}} + \bar{\bar{R}}_{12}) = \bar{\bar{\alpha}}_2^{-1} \cdot \bar{\bar{B}}_2^t \cdot \bar{\bar{T}}_{12}, \tag{34}$$

$$\bar{\bar{H}}_{1+} + \bar{\bar{H}}_{1-} \cdot \bar{\bar{R}}_{12} = \bar{\bar{H}}_{2+} \cdot \bar{\bar{T}}_{12}, \tag{35}$$

where

$$\bar{\bar{H}}_{i\pm} = \langle \bar{\bar{S}}, \quad \bar{\bar{\lambda}}_i^{-1} \cdot \bar{\bar{E}}_i \cdot \bar{\bar{N}}_{i\pm} \rangle,$$
$$\bar{\bar{\alpha}}_i^{-1} = \langle \bar{\bar{S}}, \quad \bar{\bar{E}}_i^{-1}\bar{\bar{S}}^t \rangle$$
$$\bar{\bar{\alpha}}_i^{-1} = \langle \bar{\bar{S}}, \quad \bar{\bar{E}}_i^{-1}\bar{\bar{S}}^t \rangle. \tag{36}$$

$\bar{\bar{H}}_{i\pm}$ can be rewritten as

$$\bar{\bar{H}}_{i\pm} = \bar{\bar{\Lambda}}_i^{-1} \cdot \bar{\bar{\Omega}}_{i\pm}, \tag{37}$$

where

$$\bar{\bar{\Lambda}}_i^{-1} = \langle \cdot \bar{\bar{S}}, \bar{\bar{\lambda}}_i^{-1} \cdot \bar{\bar{E}}_i \cdot \bar{\bar{S}} \rangle = \begin{bmatrix} \bar{\bar{L}}_{\epsilon i}^{-1} & 0 \\ 0 & \bar{\bar{L}}_{\mu i}^{-1} \end{bmatrix} \tag{38}$$

$\bar{\bar{L}}_{pi}$, $\bar{\bar{p}}_i$ are as defined in (12) and (13). We have simplified the above by using the identity [24]

$$\langle \bar{\bar{S}}, \bar{\bar{A}} \cdot \bar{\bar{B}} \cdot \bar{\bar{S}}^t \rangle = \langle \bar{\bar{S}}, \bar{\bar{A}} \cdot \bar{\bar{S}}^t \rangle \langle \bar{\bar{S}}, \bar{\bar{B}} \cdot \bar{\bar{S}}^t \rangle. \tag{40}$$

The above identity is true when the basis function is complete and orthonormal. It is an approximation when we use only a subset of the complete set of basis function. However, this approximation can be made arbitrarily good by using a larger basis set. Since we know $\bar{\bar{L}}_{pi}$ from (13), $\bar{\bar{\Lambda}}_i$ in (38) can be obtained by inverting $\bar{\bar{L}}_{pi}$. Of course, this is only an approximation if we have an incomplete basis set.

Solving (34) and (35), we have

$$\bar{\bar{R}}_{12} = (\bar{\bar{B}}_2^{t-1} \cdot \bar{\bar{\alpha}}_2 \cdot \bar{\bar{\alpha}}_1^{-1} \cdot \bar{\bar{B}}_1^t - \bar{\bar{H}}_{2+}^{-1} \cdot \bar{\bar{H}}_{1-})^{-1} \cdot (\bar{\bar{H}}_{2+}^{-1} \cdot \bar{\bar{H}}_{1-} - \bar{\bar{B}}_2^{t-1} \cdot \bar{\bar{\alpha}}_2 \cdot \bar{\bar{\alpha}}_1^{-1} \cdot \bar{\bar{B}}_1^t) \tag{41}$$

$$\bar{\bar{T}}_{12} = (\bar{\bar{B}}_1 \cdot \bar{\bar{\alpha}}_1^{-1} \cdot \bar{\bar{\alpha}}_2 \cdot \bar{\bar{B}}_2^{-1})^t \cdot (\bar{\bar{I}} + \bar{\bar{R}}_{12}). \tag{42}$$

With $\bar{\bar{R}}_{12}$ and $\bar{\bar{T}}_{12}$ available, we can obtain the field via (27) to (30).

The $y$-components of the fields can be derived similarly to (25) from (19) and (20).

$$\bar{A}_{1y} = \begin{bmatrix} \tilde{H}_{1y} \\ \tilde{E}_{1y} \end{bmatrix} = \bar{\bar{S}}^t(z) \cdot [\bar{\bar{G}}_{1\pm} \cdot \bar{\bar{U}}_{\pm} \cdot e^{i\bar{\bar{K}}_1|y-y'|} + \bar{\bar{G}}_{1-} \cdot e^{-i\bar{\bar{K}}_1(y-d)} \cdot \bar{\bar{R}}_{12} \cdot e^{i\bar{\bar{k}}_1|d-y'|}] \cdot \bar{F}_{1s}(z') \tag{43}$$

$$\bar{A}_{\alpha}y = \begin{bmatrix} \tilde{H}_{\alpha y} \\ \tilde{E}_{\alpha y} \end{bmatrix} = \bar{\bar{S}}^t(z) \cdot \bar{\bar{G}}_{2+} \cdot e^{i\bar{\bar{K}}_2(y-d)} \cdot \bar{\bar{T}}_{12} \cdot e^{i\bar{\bar{K}}_1|d-y'|} \cdot \bar{F}_{1s}(z') \tag{44}$$

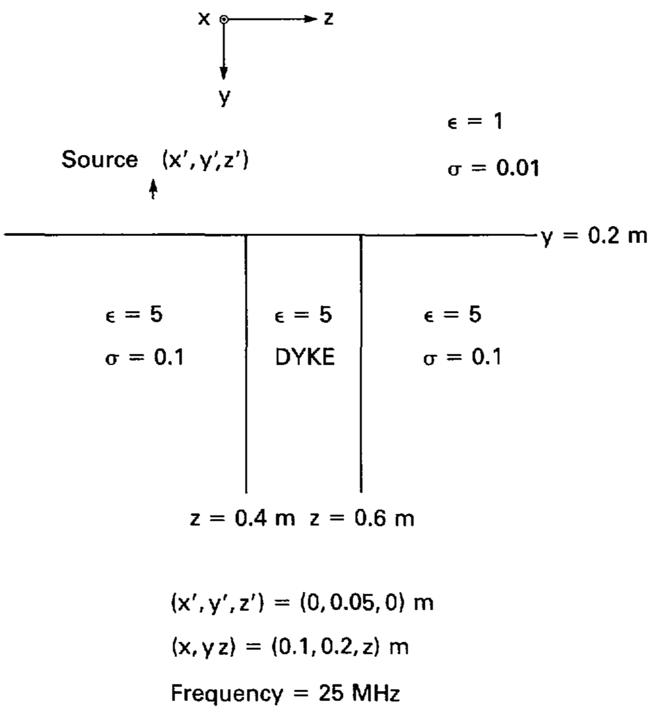


Fig. 3. The geometry assumed for the subsequent test cases.

where

$$\bar{\bar{G}}_{i\pm} - \bar{\bar{\Lambda}}_i^{-1} \cdot \begin{bmatrix} -\omega k_x \bar{\bar{\epsilon}}_i^{-1} \cdot \bar{\bar{b}}_{\epsilon_i}^t & \pm i \left\langle \bar{S}, \frac{\partial}{\partial z}\, \epsilon_i^{-1} \mu_i^{-1} \bar{S}^t \right\rangle \cdot \bar{\bar{b}}_{\mu_i}^t \cdot \bar{\bar{K}}_{\mu_i y} \\ \pm i \left\langle \bar{S}, \frac{\partial}{\partial z} - \epsilon_i^{-1} \mu_i^{-1} \bar{S}^t \right\rangle \cdot \bar{\bar{b}}_{\epsilon_i}^t \cdot \bar{\bar{K}}_{\epsilon_i y} & \omega k_x \bar{\bar{\mu}}_i^{-1} \cdot \bar{\bar{b}}_{\mu_i}^t \end{bmatrix}. \tag{45}$$

We have also simplified the above using the identity (40). Similarly, (28) and (30) can be simplified using (40) giving

$$\bar{A}_{1x} = \bar{\bar{S}}^t(z) \cdot [\bar{\bar{H}}_{1\pm} \cdot \bar{\bar{U}}_{\pm} \cdot e^{i\bar{\bar{K}}_1 |y-y'|} + \bar{\bar{H}}_{1-} \cdot e^{-i\bar{\bar{K}}_1 (y-d)} \cdot \bar{\bar{R}}_{12} \cdot e^{i\bar{\bar{K}}_1 |d-y'|}] \cdot \bar{F}_{1s}(z') \tag{46}$$

$$\bar{A}_{2x} = \bar{\bar{S}}^t(z) \cdot \bar{\bar{H}}_{2+} \cdot e^{i\bar{\bar{K}}_2 (y-d)} \cdot \bar{\bar{T}}_{12} \cdot e^{i\bar{\bar{K}}_1 |d-y'|} \cdot \bar{F}_{1s}(z'). \tag{47}$$

## RESULTS AND CONCLUSION

We have implemented the formulation above on a computer and produced some numerical results. We have experimented with Fourier spectral components, exp $(ik_{iz}z)$ as the basis functions when the half-spaces are homogeneous. In this case, the formulas can be shown to reduce to the discretized form of Sommerfeld's integral solution to the half-space problem [25], [26], i.e., the integral in Sommerfeld's solution is replaced by a summation. However, using the Fourier spectral components as basis functions gave rise to poor convergence, and we could not obtain useful results when the half-spaces are inhomogeneous and have discontinuities in the lateral direction. Because of this, we have also tried finite domain basis functions like the triangular basis functions as used in [2]. These basis functions given much better convergence. Varying grid sizes was found to give optimal convergence. The grids have finer mesh closer to the transmitter and coarser mesh far from the transmitter. The field is assumed to be zero for about two skin depths from the transmitter. Hence, the triangular basis set only spans the space $(-z_{\max}, z_{\max})$.

To test the algorithm, we have run some test cases with the geometry of Fig. 3. In the test cases, the conductivity of the dyke was chosen to be less than, equal to and greater than the conductivities next to it. When the conductivity of the dyke is equal to the conductivities of the rest of the half-space, the problem is also solvable by the Sommerfeld's half-space problem, and hence we have used the Sommerfeld's solutions [25], [26] as comparisons.

In Fig. 4, we have a line source made of vertical magnetic dipoles modulated by exp $(ik_x x)$ where $k_x = 0.5$. We plot the $H_y$ component of the reflected $H$-field from the vertically stratified half-space. The homogeneous half-space case (no dyke) compares well with Sommerfeld's solution which is obtained using the Fourier integral technique (FIT). A conductive dyke seems to increase the overall reflected signal while the tendency of the resistive dyke is to decrease it.

In Fig. 5, we have a line source of vertical electric dipole modulated by exp $(ik_x x)$ where $k_x = 0.5$, and we plot the $E_y$ component of the reflected $E$-field. Here we find that changing the conductivity of the dyke has a lesser effect (percentagewise) on the overall reflected $E$-field signal. This may be due to the manner with which the eddy currents are distributed in the

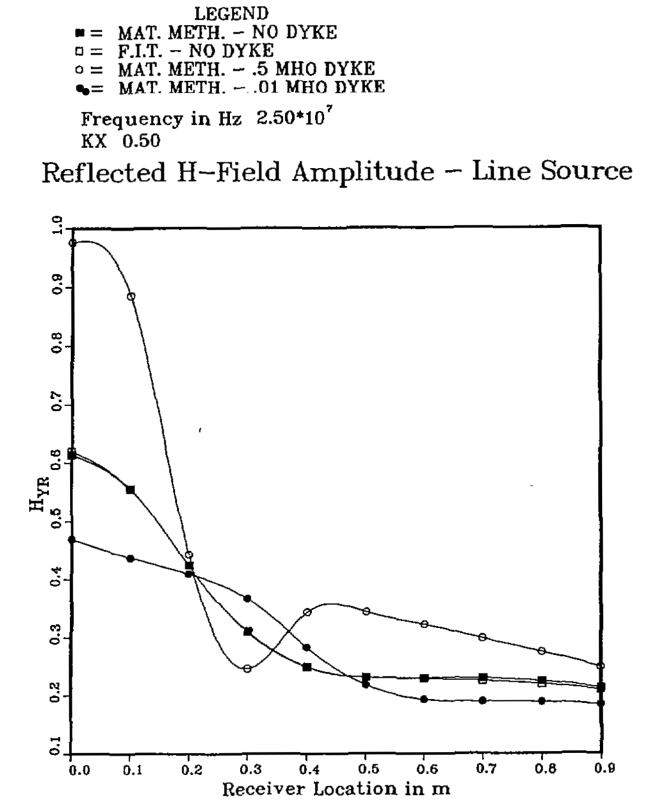


Fig. 4. Reflected $H$-field amplitude as a function of receiver location ($z$) for different dyke conductivities for a line source of vertical magnetic dipoles. The field is normalized with respect to $IA/2$ where $A$ is the area of the current loop antenna, and $I$ the current in the loop that gives the magnetic dipole moment.

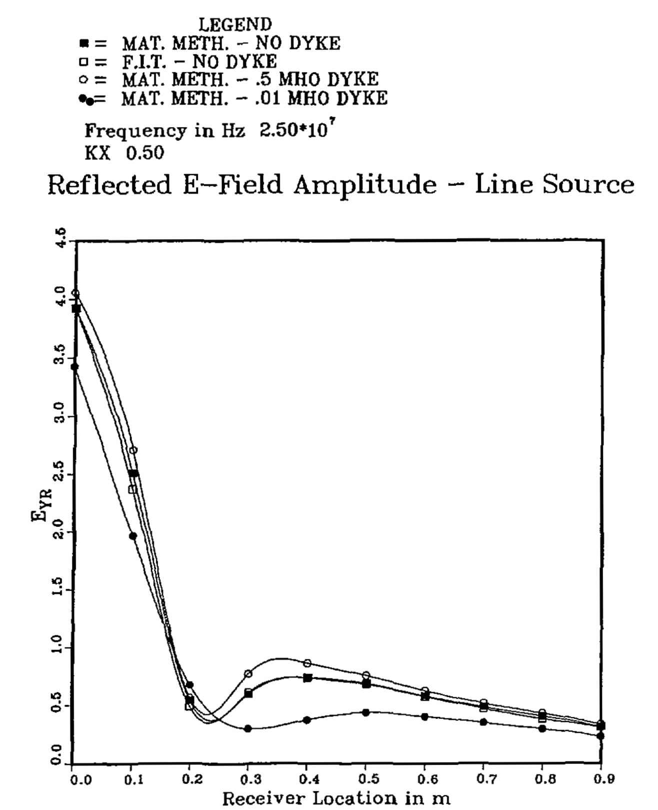


Fig. 5. Reflected $E$-field amplitude as a function of receiver location ($z$) for different dyke conductivities for a line source of vertical electric dipoles. The field is normalized with respect to $Il/(2\omega\epsilon)$.

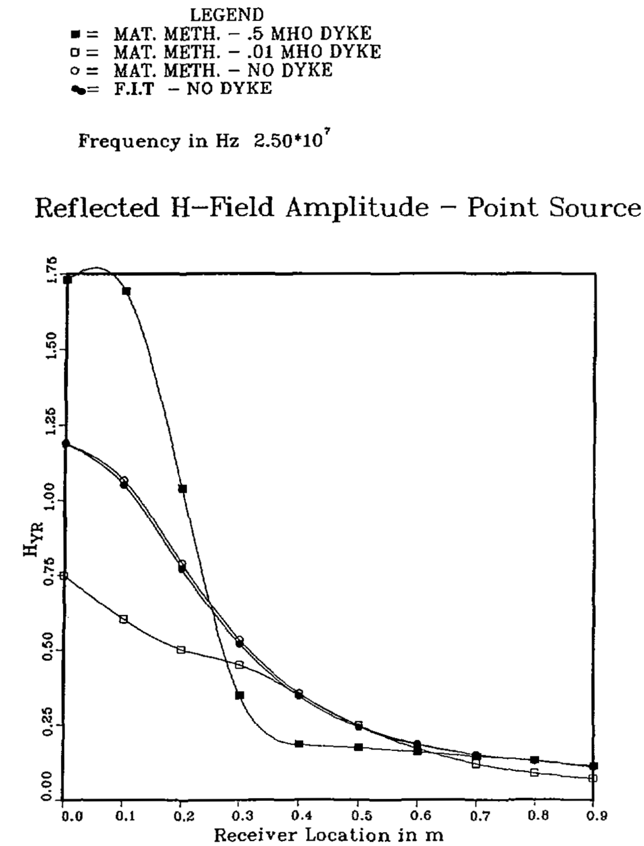


Fig. 6. Same as Fig. 4 except that the excitation source is a point source.

half-space for the different excitations by the different line sources.

In Figs. 6 and 7, we have obtained the point source response by integrating over many line sources with different exp $(ik_x x)$ dependences. The qualitative behavior due to the variation of the conductivities of the dyke is similar, and the agreement with the Fourier integral technique (Sommerfeld's solution) is excellent. For computations on a VAX 11/780 computer, the computation time was roughly an hour per curve for the above plots, while it is much faster for Figs. 4 and 5.

In conclusion, we have formulated a theory for the response of a line source or a point source on top of a vertically stratified half-space. The formulation is general, and when the right basis functions is chosen, we can reproduce Sommerfeld's half-space solution. However, in practice, we have to deal with the convergence of such a theory, since we like to work with a finite amount of computation time and finite amount of computer storage. In programming up this formulation, it was found that the assumption of losses in the two half-spaces was helpful. Further research needs to be performed to study the convergence of such a method when the medium is lossless and when the frequency is high. When the medium is lossless, we expect the basis functions to span a larger $z$-domain. Also, when the frequency is high, we expect to use finer meshes. All these imply the use of larger matrices increasing the computation time and computer storage required. However, similar problems are encountered with the finite element method. In summary, we have made a first step to outline a semi-analytic method for solving a boundary value problem which otherwise is only solvable by numerical methods like the finite element method.

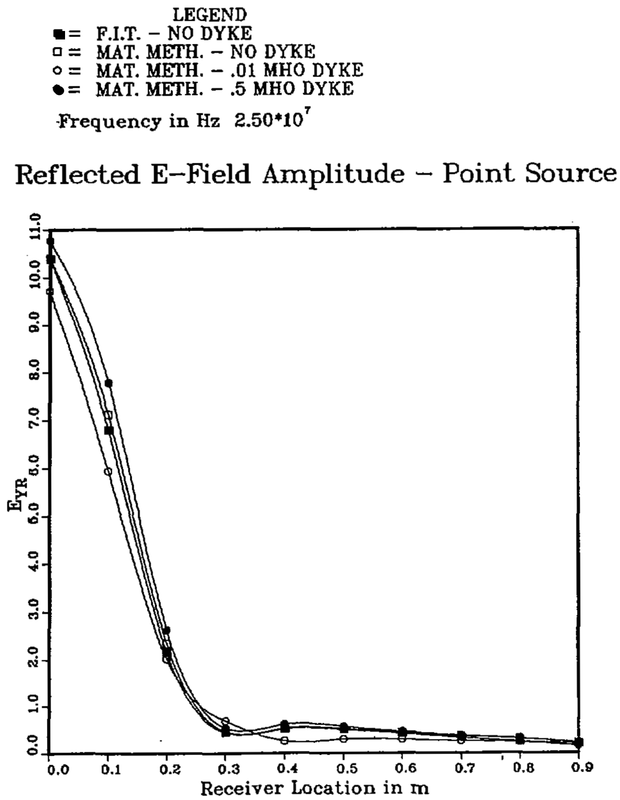


Fig. 7. Same as Fig. 5 except that the excitation source is a point source.

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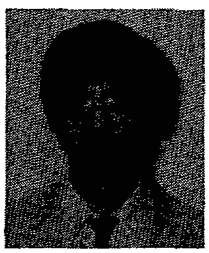

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