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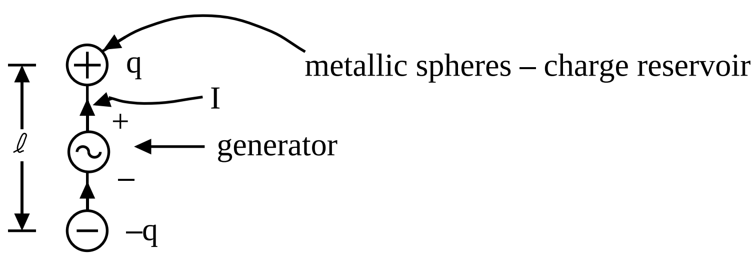
ECE 350 Lecture Notes

26. The Fields of a Hertzian Dipole

A Hertzian dipole is a dipole which is much smaller than the wavelength under construction so that we can approximate it by a point current distribution,

$$\mathbf{J}(\mathbf{r}) = \hat{z} Il\delta(\mathbf{r}). \tag{1}$$

The dipole may look like the following



l is the effective length of the dipole so that the dipole moment $p = ql$. The charge q is varying time harmonically because it is driven by the generator. Since $\frac{dq}{dt} = I$, we have

$$Il = \frac{dq}{dt} l = j\omega q l = j\omega p, \tag{2}$$

for a Hertzian dipole. We already know that the corresponding vector potential is given by

$$\mathbf{A}(\mathbf{r}) = \hat{z}\frac{\mu Il}{4\pi r} e^{-j\beta r}. \tag{3}$$

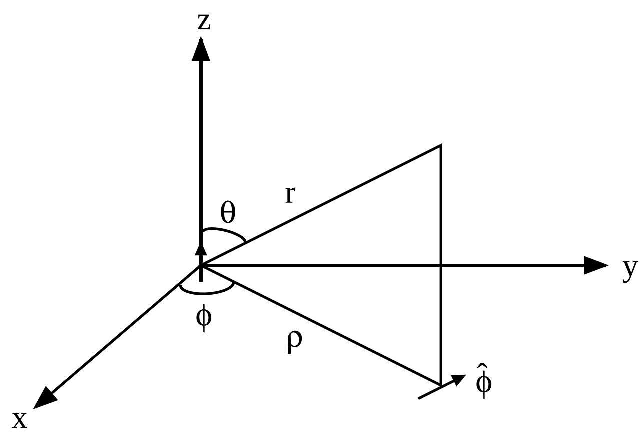
The magnetic field is obtained, using cylindrical coordinates, as

$$\mathbf{H} = \frac{1}{\mu}\nabla \times \mathbf{A} = \frac{1}{\mu}\left(\hat{\rho}\frac{1}{\rho}\frac{\partial}{\partial\phi}A_z - \hat{\phi}\frac{\partial}{\partial\rho}A_z\right), \tag{4}$$

where $\frac{\partial}{\partial\phi} = 0$, $r = \sqrt{\rho^2 + z^2}$. In the above, $\frac{\partial}{\partial\rho} = \frac{\partial r}{\partial \rho}\frac{\partial}{\partial r} = \frac{\rho}{\sqrt{\rho^2+z^2}}\frac{\partial}{\partial r} = \frac{\rho}{r}\frac{\partial}{\partial r}$.

Hence,

$$\mathbf{H} = -\hat{\phi}\frac{\rho}{r}\frac{Il}{4\pi}\left(-\frac{1}{r^2} - j\beta\frac{1}{r}\right)e^{-j\beta r}. \tag{5}$$



In spherical coordinates, $\frac{\rho}{r} = \sin\theta$, and (5) becomes

$$\mathbf{H} = \hat{\phi}\frac{Il}{4\pi r^2}(1 + j\beta r)e^{-j\beta r}\sin\theta. \tag{6}$$

The electric field can be derived using Maxwell's equations.

$$\begin{aligned}\mathbf{E} &= \frac{1}{j\omega\epsilon}\nabla\times\mathbf{H} = \frac{1}{j\omega\epsilon}\left(\hat{r}\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta H_\phi - \hat{\phi}\frac{1}{r}\frac{\partial}{\partial r}rH_\phi\right)\\ &= \frac{Ile^{-j\beta r}}{j\omega\epsilon 4\pi r^3}\left[\hat{r}2\cos\theta(1 + j\beta r) + \hat{\theta}\sin\theta(1 + j\beta r - \beta^2 r^2)\right].\end{aligned} \tag{7}$$

Case I. Near Field, $\beta r \ll 1$

$$\mathbf{E} \cong \frac{\rho}{4\pi\epsilon r^3}(\hat{r}2\cos\theta + \hat{\theta}\sin\theta), \qquad \beta r \ll 1, \tag{8}$$

$$\mathbf{H} \ll \mathbf{E}, \qquad \text{when } \beta r \ll 1. \tag{9}$$

βr could be made very small by making $\frac{r}{\lambda}$ small or by making $\omega \to 0$. The above is like the static field of a dipole.

Case II. Far Field (Radiation Field), $\beta r \gg 1$

In this case,

$$\mathbf{E} \cong \hat{\theta}j\omega\mu\frac{Il}{4\pi r}e^{-j\beta r}\sin\theta, \tag{10}$$

and

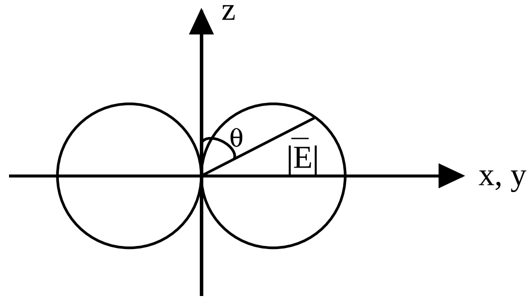
$$\mathbf{H} \cong \hat{\phi}j\beta\frac{Il}{4\pi r}e^{-j\beta r}\sin\theta. \tag{11}$$

Note that $\frac{E_\theta}{H_\phi} = \frac{\omega\mu}{\beta} = \sqrt{\frac{\mu}{\epsilon}} = \eta_0$. **E** and **H** are orthogonal to each other and are both orthogonal to the direction of propagation, i.e. as in the case of a plane wave. A spherical wave resembles a plane wave in the far field approximation.

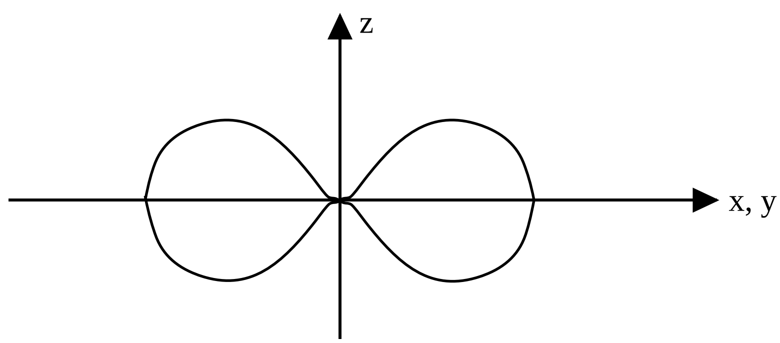
The time average power flow is given by

$$\langle \mathbf{S} \rangle = \frac{1}{2} \Re e[\mathbf{E} \times \mathbf{H}^*] = \hat{r} \frac{1}{2} \eta_0 \, |H_\phi|^2 = \hat{r} \frac{\eta_0}{2} \left(\frac{\beta I l}{4\pi r} \right)^2 \sin^2 \theta. \tag{12}$$

The **radiation field pattern** of a Hertzian dipole is the plot of $|\mathbf{E}|$ as a function of θ at a constant $\mathbf{r}$.



The **radiation power pattern** is the plot of $\langle S_r \rangle$ at a constant r.



The total power radiated by a Hertzian dipole is given by

$$P = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta r^2 \sin\theta \langle S_r \rangle = 2\pi \int_0^{\pi} d\theta \frac{\eta_0}{2} \left(\frac{\beta I l}{4\pi} \right)^2 \sin^3 \theta. \tag{13}$$

Since

$$\int_0^{\pi} d\theta \sin^3 \theta = -\int_1^{-1} (d\, \cos\theta)[1 - \cos^2 \theta] = \int_{-1}^{1} dx (1 - x^2) = \frac{4}{3}, \tag{14}$$

then

$$P = \frac{4}{3} \pi \eta_0 \left(\frac{\beta I l}{4\pi} \right)^2. \tag{15}$$

The **directive gain** of an antenna, $D(\theta, \phi)$, is defined as

$$D(\theta, \phi) = \frac{\langle S_r \rangle}{\frac{P}{4\pi r^2}}, \tag{16}$$

where $\frac{P}{4\pi r^2}$ is the power density if the power P were uniformly distributed over a sphere. Substituting (12) and (15) into the above, we have

$$D(\theta, \phi) = \frac{\frac{\eta_0}{2}\left(\frac{\beta I l}{4\pi r}\right)^2 \sin^2\theta}{\frac{1}{4\pi r^2}\frac{4}{3}\eta_0 \pi \left(\frac{\beta I l}{4\pi}\right)^2} = \frac{3}{2}\sin^2\theta. \tag{17}$$

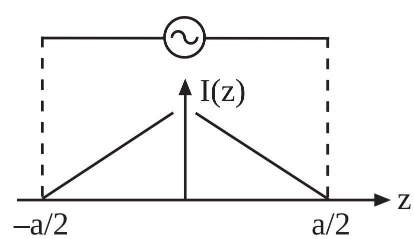
The peak of $D(\theta, \phi)$ is known as the **directivity** of an antenna. It is 1.5 in this case. If an antenna is radiating isotropically, its directivity is 1. Therefore, the lowest possible values for the directivity of an antenna is 1, whereas it can be over 100 for some antennas like reflector antennas. A **directive gain pattern** is a plot of the above function $D(\theta, \phi)$ and it resembles the radiation power pattern.

If the total power fed into the antenna instead of the total radiated power is used in the denominator of (16), the ratio is known as the **power gain** or just bf gain. The total power fed into the antenna is not equal to the total radiated power because there could be some loss in the antenna system like metallic loss.

Defining a **radiation resistance** R_r by $P = \frac{1}{2}I^2 R_r$, we have

$$R_r = \frac{2P}{I^2} = \eta_0 \left(\frac{\beta l}{6\pi}\right)^2, \qquad \text{where } \eta_0 = 377\Omega. \tag{18}$$

For example, for a Hertzian dipole with $l = 0.1\lambda$, $R_r \approx 8\Omega$. For a small dipole with no charge reservoir at the two ends, the currents have to vanish at the tip of the dipole.



The effective length of the dipole is **half** of its actual length due to the manner the currents are distributed. For example, for a half-wave dipole, $a = \frac{\lambda}{2}$, and if we use $l_{\text{eff}} = \frac{\lambda}{4}$ in (18), we have

$$R_r \approx 50\Omega. \tag{19}$$

However, a half-wave dipole is not much smaller than a wavelength and does not qualify to be a Hertzian dipole. Furthermore, the current distribution on the half-wave dipole is not triangular in shape as above. A more precise calculation shows that $R_r = 73\Omega$ for a half-wave dipole.