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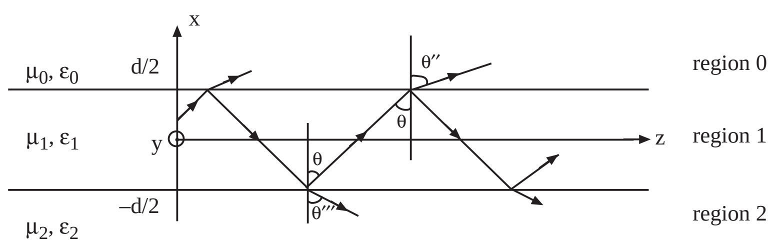
ECE 350 Lecture Notes

24. Dielectric Waveguides (Slab).

When a wave is incident from a medium with higher dielectric constant at an interface of two dielectric media, **total internal reflection** occurs when the angle of incident is larger than the **critical angle**. This fact can be used to make waves bouncing between two interfaces of a dielectric slab to be guided



Since total internal reflection occurs for both TE and TM waves, guidance is possible for both types of waves

I. TE Case $\mathbf{E} = \hat{y}E_y$

E_y is a solution to the wave equation in each region. In region 0, we assume a solution of the form

$$E_{0y} = E_0 e^{-j\beta_{0x}x - j\beta_z z}, \tag{1}$$

where

$$\beta_{0x}^2 + \beta_z^2 = \omega^2 \mu_0 \epsilon_0 = \beta_0^2. \tag{1a}$$

In region 1, we assume a solution of the form

$$E_{1y} = [A_1 e^{-j\beta_{1x}x} + B_1 e^{j\beta_{1x}x}] e^{-j\beta_z z}, \tag{2}$$

where

$$\beta_{1x}^2 + \beta_z^2 = \omega^2 \mu_1 \epsilon_1 = \beta_1^2. \tag{2a}$$

In region 2, the solution is of the form

$$E_{2y} = E_2 e^{j\beta_{2x}x - j\beta_z z}, \tag{3}$$

where

$$\beta_{2x}^2 + \beta_z^2 = \omega^2 \mu_2 \epsilon_2 = \beta_2^2. \tag{3a}$$

We assume that all the solutions in the three regions to have the same z-variation of $e^{-j\beta_z z}$ by the **phase matching** condition.

In region 1, we have an up-going wave as well as a down-going wave. The two waves have to be related by the reflection coefficient $\rho_\perp$ for the electric field at the boundaries. $\rho_\perp$ is derived earlier in the course. Therefore at $x = \frac{d}{2}$, we have

$$B_1 e^{j\beta_{1x}\frac{d}{2}} = \rho_{10\perp} A_1 e^{-j\beta_{1x}\frac{d}{2}}, \tag{4}$$

where $\rho_{10\perp}$ is the reflection coefficient at the regions 1 and 0 interface. At $x = -\frac{d}{2}$, we have

$$A_1 e^{j\beta_{1x}\frac{d}{2}} = \rho_{12\perp} B_1 e^{-j\beta_{1x}\frac{d}{2}}, \tag{5}$$

where $\rho_{12\perp}$ is the reflection coefficient at the regions 1 and 2 interface. Multiplying equations (4) and (5) together, we have,

$$A_1 B_1 e^{j\beta_{1x} d} = \rho_{12\perp}\rho_{10\perp} A_1 B_1 e^{-j\beta_{1x} d}. \tag{6}$$

A_1 and B_1 are non-zero only if

$$1 = \rho_{12\perp}\rho_{10\perp} e^{-2j\beta_{1x} d}. \tag{7}$$

The above is known as the **guidance condition** of a dielectric slab waveguide. If medium 3 is equal to medium 1, then $\rho_{12\perp} = \rho_{10\perp}$, and the guidance condition becomes

$$1 = \rho_{10\perp}^2 e^{-2j\beta_{1x} d}. \tag{8}$$

From before, for a wave incident at an angle θ,

$$\rho_{10\perp} = \frac{\eta_0 \cos\theta - \eta_1 \cos\theta''}{\eta_0 \cos\theta + \eta_1 \cos\theta''}. \tag{9}$$

Since $\beta_{1x} = \beta_1 \cos\theta$, $\beta_{0x} = \beta_0 \cos\theta''$, (9) could be written as

$$\rho_{10\perp} = \frac{\frac{\eta_0}{\beta_1}\beta_{1x} - \frac{\eta_1}{\beta_0}\beta_{0x}}{\frac{\eta_0}{\beta_1}\beta_{1x} + \frac{\eta_1}{\beta_0}\beta_{0x}} = \frac{\mu_0\beta_{1x} - \mu_1\beta_{0x}}{\mu_0\beta_{1x} + \mu_1\beta_{0x}}. \tag{10}$$

Taking the square root of (8), we have

$$\rho_{10\perp} e^{-j\beta_{1x} d} = \pm 1. \tag{11}$$

When we choose the plus sign, $B_1 = A_1$ from (4), and from (2)

$$E_{1y} = 2A_1 \cos(\beta_{1x} x) e^{-j\beta_z z} \quad \Rightarrow \text{even in x.} \tag{12}$$

When we choose the minus sign in (11) we have $B_1 = -A_1$, and

$$E_{1y} = -2jA_1 \sin(\beta_{1x} x) e^{-j\beta_z z} \quad \Rightarrow \text{odd in x.} \tag{13}$$

Multiplying (11) by $e^{j\beta_{1x}\frac{d}{2}}$ and manipulating, we have

$$\frac{\mu_0}{\mu_1}\beta_{1x}\frac{d}{2}\tan\left(\beta_{1x}\frac{d}{2}\right) = j\beta_{0x}\frac{d}{2} \qquad \text{even solutions,} \tag{14}$$

$$\frac{\mu_0}{\mu_1}\beta_{1x}\frac{d}{2}\cot\left(\beta_{1x}\frac{d}{2}\right) = j\beta_{0x}\frac{d}{2} \qquad \text{odd solutions.} \tag{15}$$

Subtracting (1a) from (2a) and solving for β_{0x}, we have

$$\beta_{0x} = [\omega^2(\mu_0\epsilon_0 - \mu_1\epsilon_1) + \beta_{1x}^2]^{\frac{1}{2}}. \tag{16}$$

In order for (14) and (15) to be satisfied, β_{0x} has to be pure imaginary. In other words, the waves in region 0 and 3 have to be evanescent and decay exponentially away from the slab. Hence

$$\beta_{0x} = -j\alpha_{0x} = -j[\omega^2(\mu_1\epsilon_1 - \mu_0\epsilon_0) - \beta_{1x}^2]^{\frac{1}{2}}, \tag{17}$$

and (14) and (15) become

$$\frac{\mu_0}{\mu_1}\beta_{1x}\frac{d}{2}\tan\beta_{1x}\frac{d}{2} = \alpha_{0x}\frac{d}{2} = \sqrt{\omega^2(\mu_1\epsilon_1 - \mu_0\epsilon_0)\frac{d^2}{4} - \left(\beta_{1x}\frac{d}{2}\right)^2} \text{ even solutions,} \tag{18}$$

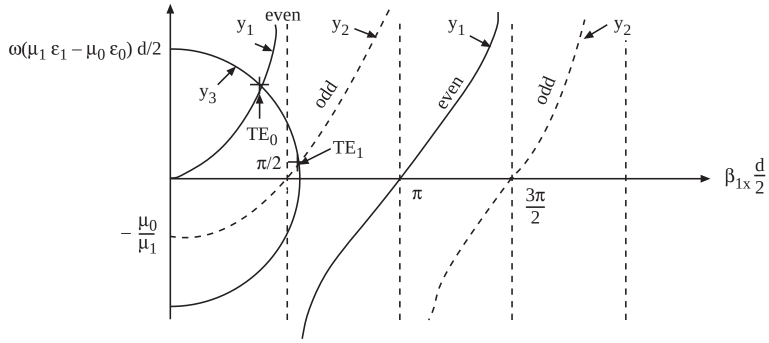
$$-\frac{\mu_0}{\mu_1}\beta_{1x}\frac{d}{2}\cot\beta_{1x}\frac{d}{2} = \alpha_{0x}\frac{d}{2} = \sqrt{\omega^2(\mu_1\epsilon_1 - \mu_0\epsilon_0)\frac{d^2}{4} - \left(\beta_{1x}\frac{d}{2}\right)^2} \text{ odd solutions.} \tag{19}$$

We can solve the above graphically by plotting

$$y_1 = \frac{\mu_0}{\mu_1}\beta_{1x}\frac{d}{2}\tan\left(\beta_{1x}\frac{d}{2}\right) \qquad \text{even solutions,} \tag{20}$$

$$y_2 = -\frac{\mu_0}{\mu_1}\beta_{1x}\frac{d}{2}\cot\left(\beta_{1x}\frac{d}{2}\right) \qquad \text{odd solutions,} \tag{21}$$

$$y_3 = \left[\omega^2(\mu_1\epsilon_1 - \mu_0\epsilon_0)\frac{d^2}{4} - \left(\beta_{1x}\frac{d}{2}\right)^2\right]^{\frac{1}{2}} = \alpha_{0x}\frac{d}{2}. \tag{22}$$



y_3 is the equation of a circle; the radius of the circle is given by

$$\omega(\mu_1\epsilon_1 - \mu_0\epsilon_0)^{\frac{1}{2}}\frac{d}{2}. \tag{23}$$

The solutions to (18) and (19) are given by the intersections of y_3 with y_1 and y_2. We note from (23) that the radius of the circle can be increased in three ways; (i) by increasing the frequency, (ii) by increasing the contrast $\frac{\mu_1\epsilon_1}{\mu_0\epsilon_0}$, and (iii) by increasing the thickness d of the slab.

When $\beta_{0x} = -j\alpha_{0x}$, the reflection coefficient is

$$\rho_{10\perp} = \frac{\mu_0\beta_{1x} + j\mu_1\alpha_{0x}}{\mu_0\beta_{1x} - j\mu_1\alpha_{0x}} = \exp\left[+2j\tan^{-1}\left(\frac{\mu_1\alpha_{0x}}{\mu_0\beta_{1x}}\right)\right], \tag{24}$$

and $|\rho_{10\perp}| = 1$. Hence there is total internal reflections and the wave is guided by total internal reflections. **Cut-off occurs** when the total internal reflection ceases to occur, i.e. when the frequency decreases such that $\alpha_{0x} = 0$. From the diagram, we see that $\alpha_{0x} = 0$ when

$$\omega(\mu_1\epsilon_1 - \mu_0\epsilon_0)^{\frac{1}{2}}\frac{d}{2} = \frac{m\pi}{2}, \qquad m = 0, 1, 2, 3, \ldots, \tag{25}$$

or

$$\omega_{mc} = \frac{m\pi}{d(\mu_1\epsilon_1 - \mu_0\epsilon_0)^{\frac{1}{2}}}, \qquad m = 0, 1, 2, 3, \ldots. \tag{26}$$

The mode that corresponds to the m-th cut-off frequency above is labeled the TE$_m$ mode. TE$_0$ mode is the mode that has no cut-off or propagates at all frequencies.

At cut-off, $\alpha_{0x} = 0$, and from (1a),

$$\beta_z = \omega\sqrt{\mu_0\epsilon_0}, \tag{27}$$

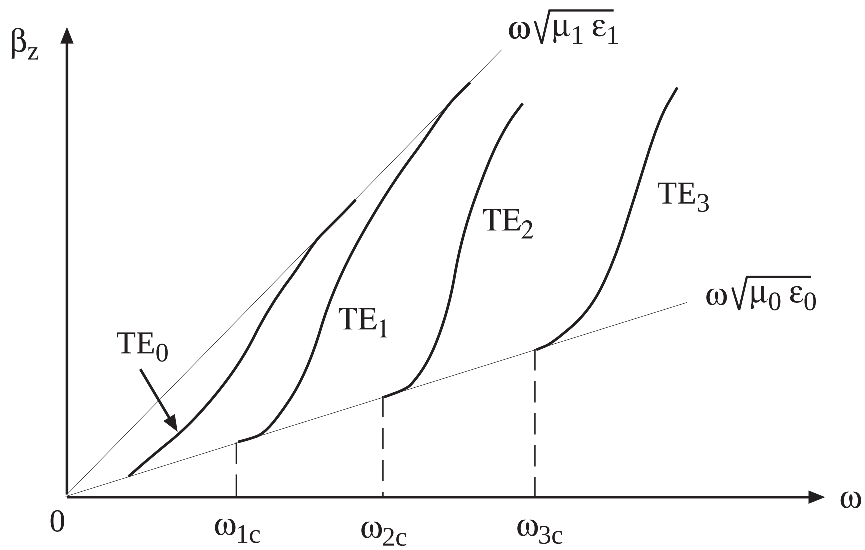
for all the modes. Hence, both the group and the phase velocities are that of the outer region. This is because when $\alpha_{0x} = 0$, the wave is not evanescent outside, and most of the energy of the mode is carried by the exterior field.

When $\omega \to \infty$, $\beta_{1x} \to \frac{n\pi}{d}$ from the diagram for all the modes. From (2a),

$$\beta_z = \sqrt{\omega^2\mu_1\epsilon_1 - \beta_{1x}^2} \approx \omega\sqrt{\mu_1\epsilon_1}, \qquad \omega \to \infty. \tag{28}$$

Hence the group and phase velocities approach that of the dielectric slab. This is because when $\omega \to \infty$, $\alpha_{0x} \to \infty$, and all the fields are trapped in the slab and propagating within it.

Because of this, the dispersion diagram of the different modes appear as below.



II. TM Case $\mathbf{H} = \hat{y} H_y$

For the TM case, a similar guidance condition analogous to (27) can be derived

$$1 = \rho_{12\|}\rho_{10\|}e^{-2j\beta_{1x}d}, \tag{29}$$

where ρ is the reflection coefficient for the TM field. Similar derivations show that the above guidance condition, for $\epsilon_2 = \epsilon_0$, $\mu_2 = \mu_0$, reduces to

$$\frac{\epsilon_0}{\epsilon_1}\beta_{1x}\frac{d}{2}\tan\beta_{1x}\frac{d}{2} = \sqrt{\omega^2(\mu_1\epsilon_1 - \mu_0\epsilon_0)\frac{d^2}{4} - \left(\beta_{1x}\frac{d}{2}\right)^2} \qquad \text{even solution,} \tag{30}$$

$$-\frac{\epsilon_0}{\epsilon_1}\beta_{1x}\frac{d}{2}\cot\beta_{1x}\frac{d}{2} = \sqrt{\omega^2(\mu_1\epsilon_1 - \mu_0\epsilon_0)\frac{d^2}{4} - \left(\beta_{1x}\frac{d}{2}\right)^2} \qquad \text{odd solution.} \tag{31}$$

Note that for equations (7) and (29), when we have two parallel metallic plates, $\rho_\| = 1$, and $\rho_\perp = \pm 1$, and the guidance condition becomes

$$1 = e^{-2j\beta_{1x}d} \quad \Rightarrow \beta_{1x} = \frac{m\pi}{d}, m = 0, 1, 2, \ldots, \tag{32}$$

which is what we have observed before.