24. Dielectric Waveguides (Slab).

When a wave is incident from a medium with higher dielectric constant at an interface of two dielectric media, **total internal reflection** occurs when the angle of incident is larger than the **critical angle**. This fact can be used to make waves bouncing between two interfaces of a dielectric slab to be guided

Since total internal reflection occurs for both TE and TM waves, guidance is possible for both types of waves

**I. TE Case** \( E = \hat{y}E_y \)

\( E_y \) is a solution to the wave equation in each region. In region 0, we assume a solution of the form

\[
E_{0y} = E_0 e^{-j\beta_0 x - j\beta_0 z},
\]

where

\[
\beta_0^2 + \beta_0^2 = \omega^2 \mu_0 \varepsilon_0 = \beta_0^2. \tag{1a}
\]

In region 1, we assume a solution of the form

\[
E_{1y} = [A_1 e^{-j\beta_{1x} x} + B_1 e^{j\beta_{1x} x}] e^{-j\beta_1 z},
\]

where

\[
\beta_{1x}^2 + \beta_1^2 = \omega^2 \mu_1 \varepsilon_1 = \beta_1^2. \tag{2a}
\]

In region 2, the solution is of the form

\[
E_{2y} = E_2 e^{j\beta_{2x} x - j\beta_2 z},
\]

where

\[
\beta_{2x}^2 + \beta_2^2 = \omega^2 \mu_2 \varepsilon_2 = \beta_2^2. \tag{3a}
\]
We assume that all the solutions in the three regions to have the same $z$-variation of $e^{-j\beta z}$ by the **phase matching** condition.

In region 1, we have an up-going wave as well as a down-going wave. The two waves have to be related by the reflection coefficient $\rho_\perp$ for the electric field at the boundaries. $\rho_\perp$ is derived earlier in the course. Therefore at $x = \frac{d}{2}$, we have

$$B_1 e^{j\beta_1 x} = \rho_{10\perp} A_1 e^{-j\beta_1 x}, \quad (4)$$

where $\rho_{10\perp}$ is the reflection coefficient at the regions 1 and 0 interface. At $x = -\frac{d}{2}$, we have

$$A_1 e^{j\beta_1 x} = \rho_{12\perp} B_1 e^{-j\beta_1 x}, \quad (5)$$

where $\rho_{12\perp}$ is the reflection coefficient at the regions 1 and 2 interface. Multiplying equations (4) and (5) together, we have,

$$A_1 B_1 e^{j\beta_1 x} = \rho_{12\perp} \rho_{10\perp} A_1 B_1 e^{-j\beta_1 x}. \quad (6)$$

$A_1$ and $B_1$ are non-zero only if

$$1 = \rho_{12\perp} \rho_{10\perp} e^{-2j\beta_1 x}. \quad (7)$$

The above is known as the **guidance condition** of a dielectric slab waveguide. If medium 3 is equal to medium 1, then $\rho_{12\perp} = \rho_{10\perp}$, and the guidance condition becomes

$$1 = \rho_{10\perp}^2 e^{-2j\beta_1 x}. \quad (8)$$

From before, for a wave incident at an angle $\theta$,

$$\rho_{10\perp} = \frac{\eta_0 \cos \theta - \eta_1 \cos \theta^n}{\eta_0 \cos \theta + \eta_1 \cos \theta^n}. \quad (9)$$

Since $\beta_{lx} = \beta_1 \cos \theta$, $\beta_{0x} = \beta_0 \cos \theta^n$, (9) could be written as

$$\rho_{10\perp} = \frac{\eta_0 \beta_{lx} - \eta_1 \beta_{0x}}{\eta_0 \beta_{lx} + \eta_1 \beta_{0x}} = \frac{\mu_0 \beta_{lx} - \mu_1 \beta_{0x}}{\mu_0 \beta_{lx} + \mu_1 \beta_{0x}}. \quad (10)$$

Taking the square root of (8), we have

$$\rho_{10\perp} e^{-j\beta_1 x} = \pm 1. \quad (11)$$

When we choose the plus sign, $B_1 = A_1$ from (4), and from (2)

$$E_{1y} = 2A_1 \cos(\beta_{lx} x) e^{-j\beta z} \Rightarrow \text{even in } x. \quad (12)$$

When we choose the minus sign in (11) we have $B_1 = -A_1$, and

$$E_{1y} = -2j A_1 \sin(\beta_{lx} x) e^{-j\beta z} \Rightarrow \text{odd in } x. \quad (13)$$
Multiplying (11) by $e^{j\beta_2 x}$ and manipulating, we have

$$\frac{\mu_0}{\mu_1} \beta_{1x} \frac{d}{2} \tan \left( \beta_{1x} \frac{d}{2} \right) = j\beta_{0x} \frac{d}{2} \quad \text{even solutions}, \quad (14)$$

$$\frac{\mu_0}{\mu_1} \beta_{1x} \frac{d}{2} \cot \left( \beta_{1x} \frac{d}{2} \right) = j\beta_{0x} \frac{d}{2} \quad \text{odd solutions}. \quad (15)$$

Subtracting (1a) from (2a) and solving for $\beta_{0x}$, we have

$$\beta_{0x} = \sqrt{\omega^2 (\mu_0 \varepsilon_0 - \mu_1 \varepsilon_1) - \beta_{1x}^2} \frac{d}{2}. \quad (16)$$

In order for (14) and (15) to be satisfied, $\beta_{0x}$ has to be pure imaginary. In other words, the waves in region 0 and 3 have to be evanescent and decay exponentially away from the slab. Hence

$$\beta_{0x} = -j \alpha_{0x} = -j \sqrt{\omega^2 (\mu_1 \varepsilon_1 - \mu_0 \varepsilon_0) - \beta_{1x}^2} \frac{d}{2}. \quad (17)$$

and (14) and (15) become

$$\frac{\mu_0}{\mu_1} \beta_{1x} \frac{d}{2} \tan \beta_{1x} \frac{d}{2} = \alpha_{0x} \frac{d}{2} = \sqrt{\omega^2 (\mu_1 \varepsilon_1 - \mu_0 \varepsilon_0) \frac{d^2}{4} - \left( \beta_{1x} \frac{d}{2} \right)^2} \quad \text{even solutions}, \quad (18)$$

$$-\frac{\mu_0}{\mu_1} \beta_{1x} \frac{d}{2} \cot \beta_{1x} \frac{d}{2} = \alpha_{0x} \frac{d}{2} = \sqrt{\omega^2 (\mu_1 \varepsilon_1 - \mu_0 \varepsilon_0) \frac{d^2}{4} - \left( \beta_{1x} \frac{d}{2} \right)^2} \quad \text{odd solutions}. \quad (19)$$

We can solve the above graphically by plotting

$$y_1 = \frac{\mu_0}{\mu_1} \beta_{1x} \frac{d}{2} \tan \left( \beta_{1x} \frac{d}{2} \right) \quad \text{even solutions}, \quad (20)$$

$$y_2 = -\frac{\mu_0}{\mu_1} \beta_{1x} \frac{d}{2} \cot \left( \beta_{1x} \frac{d}{2} \right) \quad \text{odd solutions}, \quad (21)$$

$$y_3 = \sqrt{\omega^2 (\mu_1 \varepsilon_1 - \mu_0 \varepsilon_0) \frac{d^2}{4} - \left( \beta_{1x} \frac{d}{2} \right)^2} = \alpha_{0x} \frac{d}{2}. \quad (22)$$
\( y_3 \) is the equation of a circle; the radius of the circle is given by

\[
\omega (\mu_1 \epsilon_1 - \mu_0 \epsilon_0)^{\frac{1}{2}} \frac{d}{2}.
\]  

(23)

The solutions to (18) and (19) are given by the intersections of \( y_0 \) with \( y_1 \) and \( y_2 \). We note from (23) that the radius of the circle can be increased in three ways; (i) by increasing the frequency, (ii) by increasing the contrast \( \frac{\mu_1}{\mu_0} \), and (iii) by increasing the thickness \( d \) of the slab.

When \( \beta_{0x} = -j \alpha_{0x} \), the reflection coefficient is

\[
\rho_{10\perp} = \frac{\mu_0 \beta_{1x} + j \mu_1 \alpha_{0x}}{\mu_0 \beta_{1x} - j \mu_1 \alpha_{0x}} = \exp \left[ +2j \tan^{-1} \left( \frac{\mu_1 \alpha_{0x}}{\mu_0 \beta_{1x}} \right) \right],
\]  

(24)

and \(|\rho_{10\perp}| = 1\). Hence there is total internal reflections and the wave is guided by total internal reflections. **Cut-off occurs** when the total internal reflection ceases to occur, i.e. when the frequency decreases such that \( \alpha_{0x} = 0 \).

From the diagram, we see that \( \alpha_{0x} = 0 \) when

\[
\omega (\mu_1 \epsilon_1 - \mu_0 \epsilon_0)^{\frac{1}{2}} \frac{d}{2} = \frac{m\pi}{2}, \quad m = 0, 1, 2, 3, \ldots,
\]  

(25)

or

\[
\omega_{mc} = \frac{m\pi}{d(\mu_1 \epsilon_1 - \mu_0 \epsilon_0)^{\frac{1}{2}}}, \quad m = 0, 1, 2, 3, \ldots.
\]  

(26)

The mode that corresponds to the \( m \)-th cut-off frequency above is labeled the TE\(_m\) mode. TE\(_0\) mode is the mode that has no cut-off or propagates at all frequencies.

At cut-off, \( \alpha_{0x} = 0 \), and from (1a),

\[
\beta_z = \omega \sqrt{\mu_0 \epsilon_0},
\]  

(27)

for all the modes. Hence, both the group and the phase velocities are that of the outer region. This is because when \( \alpha_{0x} = 0 \), the wave is not evanescent outside, and most of the energy of the mode is carried by the exterior field.

When \( \omega \to \infty \), \( \beta_{1x} \to \frac{n\pi}{d} \) from the diagram for all the modes. From (2a),

\[
\beta_z = \sqrt{\omega^2 \mu_1 \epsilon_1 - \beta_{1x}^2} \approx \omega \sqrt{\mu_1 \epsilon_1}, \quad \omega \to \infty.
\]  

(28)

Hence the group and phase velocities approach that of the dielectric slab. This is because when \( \omega \to \infty \), \( \alpha_{0x} \to \infty \), and all the fields are trapped in the slab and propagating within it.

Because of this, the dispersion diagram of the different modes appear as below.
II. TM Case $H = \hat{y}H_y$

For the TM case, a similar guidance condition analogous to (27) can be derived

$$1 = \rho_{12||\rho_{10||} e^{-2j\beta_1 xd},}$$

where $\rho$ is the reflection coefficient for the TM field. Similar derivations show that the above guidance condition, for $\varepsilon_2 = \varepsilon_0$, $\mu_2 = \mu_0$, reduces to

$$\frac{\varepsilon_0}{\varepsilon_1} \beta_{1x} \frac{d}{2} \tan \frac{\beta_{1x}}{2} = \sqrt{\omega^2 (\mu_1 \varepsilon_1 - \mu_0 \varepsilon_0) \frac{d^2}{4} - \left( \beta_{1x} \frac{d}{2} \right)^2}$$

even solution,

$$\frac{-\varepsilon_0}{\varepsilon_1} \beta_{1x} \frac{d}{2} \cot \frac{\beta_{1x}}{2} = \sqrt{\omega^2 (\mu_1 \varepsilon_1 - \mu_0 \varepsilon_0) \frac{d^2}{4} - \left( \beta_{1x} \frac{d}{2} \right)^2}$$

odd solution.

Note that for equations (7) and (29), when we have two parallel metallic plates, $\rho_{\parallel} = 1$, and $\rho_{\perp} = \pm 1$, and the guidance condition becomes

$$1 = e^{-2j\beta_{1x} d} \Rightarrow \beta_{1x} = \frac{m\pi}{d}, m = 0, 1, 2, \ldots,$$

which is what we have observed before.