22. Hollow Waveguide.

A hollow cylindrical waveguide of uniform and arbitrary cross-section can guide waves. The fields inside a hollow waveguide can guide waves of both TE and TM types. When the field is of TE type, the electric field is purely transverse to the direction of wave propagation \( z \); Hence \( E_z = 0 \). For TM fields, the magnetic field is purely transverse to the \( z \)-axis and hence, \( H_z = 0 \). Therefore, the field components of TE fields are

\[
E_x, E_y, H_x, H_y, H_z,
\]

and for TM fields, they are

\[
H_x, H_y, E_x, E_y, E_z.
\]

We can hence characterize TE fields as having \( E_z = 0, H_z \neq 0 \), and TM fields as \( H_z = 0, E_z \neq 0 \). Hence, the \( z \)-component of the \( H \) field can be used to characterize TE fields, while the \( z \)-component of the \( E \) field can be used to characterize TM fields in a hollow waveguide. Given \( E_z \) and \( H_z \), it will be desirable to derive the transverse components of the fields. We shall denote a vector transverse to \( \hat{z} \) by a subscript \( s \). In this notation, Maxwell’s equations become

\[
\left( \nabla_s + \hat{z} \frac{\partial}{\partial z} \right) \times (\mathbf{H}_s + \hat{z}H_z) = j\omega \varepsilon (\mathbf{E}_s + \hat{z}E_z),
\]

(1)

\[
\left( \nabla_s + \hat{z} \frac{\partial}{\partial z} \right) \times (\mathbf{E}_s + \hat{z}E_z) = -j\omega \mu (\mathbf{H}_s + \hat{z}H_z),
\]

(2)

where \( \nabla_s = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} \), and \( \mathbf{E}_s \) and \( \mathbf{H}_s \) are the electric field and the magnetic field, respectively, transverse to the \( z \) direction. Equating the transverse components in (1) and (2), we have

\[
\nabla_s \times \hat{z}H_z + \frac{\partial}{\partial z} \hat{z} \times \mathbf{H}_s = j\omega \varepsilon \mathbf{E}_s,
\]

(3)

\[
\nabla_s \times \hat{z}E_z + \frac{\partial}{\partial z} \hat{z} \times \mathbf{E}_s = -j\omega \mu \mathbf{H}_s.
\]

(4)
Substituting (4) for \( \mathbf{H}_s \) into (3), we have

\[
\nabla_s \times \hat{z}H_z + \frac{\partial}{\partial z} \hat{z} \left( \nabla_s \times \hat{z}E_z + \frac{\partial}{\partial z} \hat{z} \times \mathbf{E}_s \right) = j\omega \varepsilon \mathbf{E}_s. \tag{5}
\]

Using the vector identity

\[
\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B} (\mathbf{A} \cdot \mathbf{C}) - \mathbf{C} (\mathbf{A} \cdot \mathbf{B}), \tag{6}
\]

we can show that

\[
\hat{z} \times \nabla_s \times \hat{z}E_z = \nabla_s (\hat{z} \cdot \hat{z}E_z) - \hat{z}E_z (\hat{z} \cdot \nabla_s) = \nabla_s E_z, \tag{7}
\]

and

\[
\hat{z} \times (\hat{z} \times \mathbf{E}_s) = \hat{z}(\hat{z} \cdot \mathbf{E}_s) - \mathbf{E}_s (\hat{z} \cdot \hat{z}) = -\mathbf{E}_s. \tag{8}
\]

Hence, (5) becomes

\[
\nabla_s \times \hat{z}H_z + \frac{j}{\omega \mu} \frac{\partial}{\partial z} \nabla_s E_z - \frac{j}{\omega \mu} \frac{\partial^2}{\partial z^2} \mathbf{E}_s = j\omega \varepsilon \mathbf{E}_s. \tag{9}
\]

If \( \mathbf{E} \) is of the form \( \mathbf{A} e^{-j\beta_z z} + \mathbf{B} e^{j\beta_z z} \), then \( \frac{\partial^2}{\partial z^2} = -\beta^2_z \) and (9) becomes

\[
\mathbf{E}_s = \frac{1}{\omega^2 \mu \varepsilon - \beta^2_z} \left[ \frac{\partial}{\partial z} \nabla_s E_z - j\omega \mu \nabla_s \times \hat{z}H_z \right]. \tag{10}
\]

In a similar fashion, we obtain

\[
\mathbf{H}_s = \frac{1}{\omega^2 \mu \varepsilon - \beta^2_z} \left[ \frac{\partial}{\partial z} \nabla_s H_z + j\omega \varepsilon \nabla_s \times \hat{z}E_z \right]. \tag{11}
\]

The above equations can be used to derive the transverse components of the fields given the \( \hat{z} \)-components. Hence, in general, we only need to know the \( \hat{z} \)-components of the fields.

**I. Rectangular Waveguides**

Rectangular waveguides are a special case of cylindrical waveguides with uniform rectangular cross section. Hence, we can divide the waves inside the waveguide into TM and TE types.

![Rectangular Waveguide Diagram](image-url)
**TM Case, $H_z = 0, E_z \neq 0$**

Inside the waveguide, we have a source free region, therefore

$$[\nabla^2 + \omega^2 \mu \varepsilon]E = 0, \quad (12)$$

or

$$[\nabla^2 + \omega^2 \mu \varepsilon]E_z = 0. \quad (13)$$

Equation (13) admits solutions of the form

$$E_z = E_0 \left\{ \frac{\sin \beta_x x}{\cos \beta_x x} \right\} \left\{ \frac{\sin \beta_y y}{\cos \beta_y y} \right\} e^{-j\beta_z z}, \quad (14)$$

since

$$\frac{\partial^2}{\partial x^2} \left\{ \frac{\sin \beta_x x}{\cos \beta_x x} \right\} = \beta_x^2 \left\{ \frac{\sin \beta_x x}{\cos \beta_x x} \right\}, \quad (15)$$

$$\frac{\partial^2}{\partial y^2} \left\{ \frac{\sin \beta_y y}{\cos \beta_y y} \right\} = -\beta_y^2 \left\{ \frac{\sin \beta_y y}{\cos \beta_y y} \right\}, \quad \frac{\partial^2}{\partial z^2} e^{-j\beta_z z} = -\beta_z^2 e^{j\beta_z z}. \quad (16)$$

Therefore

$$(\nabla^2 + \omega^2 \mu \varepsilon)E_z = (-\beta_x^2 - \beta_y^2 - \beta_z^2 + \omega^2 \mu \varepsilon)E_z = 0. \quad (17)$$

This is only possible if

$$\beta_x^2 + \beta_y^2 + \beta_z^2 = \omega^2 \mu \varepsilon, \quad (18)$$

which is the dispersion relation. The boundary conditions require that

$$E_z(x = 0) = 0, \quad E_z(y = 0) = 0. \quad (19)$$

Hence, the admissible solution is

$$E_z = E_0 \sin(\beta_x x) \sin(\beta_y y) e^{-j\beta_z z}. \quad (20)$$

Also, we require that

$$E_z(x = a) = 0, \quad E_z(y = b) = 0. \quad (21)$$

This is only possible if $\sin(\beta_x a) = 0$ and $\sin(\beta_y b) = 0$, or

$$\beta_x a = m\pi, m = 0, 1, 2, \ldots, \quad \beta_y b = n\pi, n = 0, 1, 2, 3, \ldots. \quad (22)$$

However, when $m$ or $n = 0$, $E_z = 0$. Hence, we have

$$\beta_x = \frac{m\pi}{a}, \quad m \geq 1, \quad \beta_y = \frac{n\pi}{b}, \quad n \geq 1, \quad (23)$$
which are the **guidance conditions**. To get the transverse \( E \) and \( H \) fields, we use (10) and (11)

\[
\begin{align*}
E_x &= \frac{1}{\omega^2 \mu \epsilon - \beta_z^2} \frac{\partial}{\partial z} E_z = -j \beta_x \beta_z \frac{E_0}{\beta_z^2 + \beta_y^2} \cos(\beta_x x) \sin(\beta_y y) e^{-j \beta_z z}, \\
E_y &= \frac{1}{\omega^2 \mu \epsilon - \beta_z^2} \frac{\partial}{\partial y} E_z = -j \beta_x \beta_z \frac{E_0}{\beta_z^2 + \beta_y^2} \sin(\beta_x x) \cos(\beta_y y) e^{-j \beta_z z}, \\
H_x &= \frac{j \omega \epsilon}{\omega^2 \mu \epsilon - \beta_z^2} \frac{\partial}{\partial y} E_z = \frac{j \omega \epsilon \beta_y}{\beta_z^2 + \beta_y^2} \cos(\beta_x x) \cos(\beta_y y) e^{-j \beta_z z}, \\
H_y &= -\frac{j \omega \epsilon}{\omega^2 \mu \epsilon - \beta_z^2} \frac{\partial}{\partial x} E_z = -\frac{j \omega \epsilon \beta_x}{\beta_z^2 + \beta_y^2} \cos(\beta_x x) \sin(\beta_y y) e^{-j \beta_z z}.
\end{align*}
\]

We note that the electric fields satisfy their boundary conditions. From the dispersion relation (18), we have

\[
\beta_z = \sqrt{\omega^2 \mu \epsilon - \left( \frac{m \pi}{a} \right)^2 - \left( \frac{n \pi}{b} \right)^2}.
\]

The solution that corresponds to a particular choice of \( m \) and \( n \) in (23) is known as the \textbf{TM}_{mn} mode. For a given TM_{mn} mode, \( \beta_z \) will be pure imaginary if

\[
\omega^2 \mu \epsilon < \left( \frac{m \pi}{a} \right)^2 + \left( \frac{n \pi}{b} \right)^2,
\]

or

\[
\omega < \frac{1}{\sqrt{\mu \epsilon}} \left[ \left( \frac{m \pi}{a} \right)^2 + \left( \frac{n \pi}{b} \right)^2 \right]^{\frac{1}{2}}.
\]

In this case, the mode is cutoff, and the fields decay in the \( z \)-direction and become purely \textbf{evanescent}. We define the cutoff frequency for the TM_{mn} mode to be

\[
\omega_{mnc} = \frac{1}{\sqrt{\mu \epsilon}} \left[ \left( \frac{m \pi}{a} \right)^2 + \left( \frac{n \pi}{b} \right)^2 \right]^{\frac{1}{2}} = v \left[ \left( \frac{m \pi}{a} \right)^2 + \left( \frac{n \pi}{b} \right)^2 \right]^{\frac{1}{2}}.
\]

The TM_{mn} mode will not propagate if

\[
\omega < \omega_{mnc} \text{ or } f < f_{mnc},
\]

where \( f_{mnc} = \frac{\omega_{mnc}}{2\pi} \). The corresponding cutoff wavelength is

\[
\lambda_{mnc} = 2\pi \left[ \left( \frac{m \pi}{a} \right)^2 + \left( \frac{n \pi}{b} \right)^2 \right]^{-\frac{1}{2}}.
\]

Only when the wavelength \( \lambda \) is smaller than this “size” can the wave “enter” the waveguide and be guided as the TM_{mn} mode.
To find the power flowing in the waveguide, we use the Poynting theorem.

\[
S_z = E_x H_y^* - E_y H_x^*,
\]

\[
= \frac{\omega \varepsilon \beta_y^2}{(\beta_x^2 + \beta_y^2)^2} |E_0|^2 \cos^2(\beta_x x) \sin^2(\beta_y y) + \frac{\omega \varepsilon \beta_x^2}{(\beta_x^2 + \beta_y^2)^2} |E_0|^2 \sin^2(\beta_x x) \cos^2(\beta_y y)
\]

\[
= \frac{\omega \varepsilon \beta_z}{(\beta_x^2 + \beta_y^2)^2} |E_0|^2 [\beta_x^2 \cos^2(\beta_x x) \sin^2(\beta_y y) + \beta_y^2 \sin^2(\beta_x x) \cos^2(\beta_y y)].
\]

The total power

\[
P_z = \int_0^b dy \int_0^a dx S_z = \frac{\omega \varepsilon \beta_z ab |E_0|^2}{4(\beta_x^2 + \beta_y^2)^2} (\beta_x^2 + \beta_y^2) = \frac{\omega \varepsilon \beta_z ab |E_0|^2}{4(\beta_x^2 + \beta_y^2)}. \tag{35}
\]

When \( f < f_{mce} \), \( \beta_z \) is purely imaginary and the power becomes purely reactive. No real power or time average power flows down a waveguide when all the modes are cutoff.

**TE Case, \( E_z = 0, H_z \neq 0 \).**

In this case,

\[
H_z = H_0 \cos(\beta_x x) \cos(\beta_y y) e^{-j\beta_z z},
\]

so that from equations (10) and (11), we have,

\[
E_x = -\frac{j \omega \mu}{\omega^2 \mu - \beta_z^2} \frac{\partial}{\partial y} H_z = \frac{j \omega \mu \beta_y}{\beta_x^2 + \beta_y^2} H_0 \cos(\beta_x x) \sin(\beta_y y) e^{-j\beta_z z},
\]

\[
E_y = \frac{j \omega \mu}{\omega^2 \mu - \beta_z^2} \frac{\partial}{\partial x} H_z = -\frac{j \omega \mu \beta_x}{\beta_x^2 + \beta_y^2} H_0 \sin(\beta_x x) \cos(\beta_y y) e^{-j\beta_z z},
\]

\[
H_x = \frac{1}{\omega^2 \mu - \beta_z^2} \frac{\partial}{\partial z} \frac{\partial}{\partial x} H_z = \frac{j \beta_x \beta_z}{\beta_x^2 + \beta_y^2} H_0 \sin(\beta_x x) \cos(\beta_y y) e^{-j\beta_z z},
\]

\[
H_y = \frac{1}{\omega^2 \mu - \beta_z^2} \frac{\partial}{\partial z} \frac{\partial}{\partial y} H_z = \frac{j \beta_y \beta_z}{\beta_x^2 + \beta_y^2} H_0 \cos(\beta_x x) \sin(\beta_y y) e^{-j\beta_z z},
\]

where \( \beta_x^2 + \beta_y^2 + \beta_z^2 = \beta^2 = \omega^2 \mu \varepsilon \). Matching boundary conditions for the tangential electric field requires that

\[
\beta_x = \frac{m \pi}{a}, m = 0, 1, 2, 3, \ldots, \quad \beta_y = \frac{n \pi}{b}, n = 0, 1, 2, 3, \ldots.
\]

Unlike the TM case, the TE case can have either \( m \) or \( n \) equal to zero. Hence, TE_{m0} or TE_{0n} modes exist. However, when both \( m \) and \( n \) are zero, \( H_z = H_0 e^{-j\beta_z z}, H_x = H_y = 0 \), and \( \nabla \cdot \mathbf{H} \neq 0 \), therefore, TE_{00} mode cannot exist.

For the TE_{mn} modes, the subscript \( m \) is associated with the longer side of the rectangular waveguide, while \( n \) is associated with the shorter side. In
the case of \( \text{TE}_{m0} \) mode, \( \beta_y = 0 \), implying that \( E_x = 0, E_y \neq 0, H_y = 0, H_x \neq 0, H_z \neq 0 \). The fields resemble that of the \( \text{TE}_{m} \) mode in a parallel plate waveguide. For the general \( \text{TE}_{mn} \) mode, the dispersion relation is

\[
\beta_z = \sqrt{\omega^2 \mu \varepsilon - \left( \frac{m\pi}{a} \right)^2 - \left( \frac{n\pi}{b} \right)^2}.
\]

(42)

Hence, the \( \text{TE}_{mn} \) mode and the \( \text{TM}_{mn} \) mode have the same cutoff frequency and they are degenerate.

**Example: Designing a Waveguide to Propagate only the \( \text{TE}_{10} \) mode**

The cutoff frequency of a \( \text{TM}_{mn} \) or a \( \text{TE}_{mn} \) mode is given by

\[
\omega_{mnc} = \frac{1}{\sqrt{\mu \varepsilon}} \left[ \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 \right]^{\frac{1}{2}}.
\]

(43)

Usually, \( a \) is assumed to be larger than \( b \) so that \( \text{TE}_{10} \) mode has the lowest cutoff frequency, which is given by

\[
f_{10c} = \frac{v}{2a} \quad \text{or} \quad \lambda_{10c} = 2a,
\]

(44)

where \( v = \frac{1}{\sqrt{\mu \varepsilon}} \), and \( f_{10c} = \omega_{10c}^{\frac{1}{2}} \). The next higher cutoff frequency is either \( f_{20c} \) or \( f_{01c} \) depending on the ratio of \( a \) to \( b \).

\[
f_{20c} = \frac{v}{a}, \quad f_{01c} = \frac{v}{2b}.
\]

(45)

If \( a > 2b \), \( f_{20c} < f_{01c} \), and if \( a < 2b \), \( f_{20c} > f_{01c} \). \( f_{20c} = f_{01c} \) if \( a = 2b \). When \( a = 2b \), and we want a waveguide to carry only the \( \text{TE}_{10} \) mode between 10 GHz and 20 GHz. Therefore, we want \( f_{10c} = 10 \) GHz, and \( f_{20c} = f_{01c} = 20 \) GHz. If the waveguide is filled with air, then \( v = 3 \times 10^8 \frac{m}{s} \), and we deduce that

\[
a = \frac{v}{2f_{10c}} = 1.5 \text{cm}, \quad b = \frac{v}{2f_{01c}} = 0.75.
\]

(46)

In such a rectangular waveguide, only the \( \text{TE}_{10} \) will propagate above 10 GHz and below 20 GHz. The other modes are all cutoff. Note that no mode could propagate below 10 GHz.