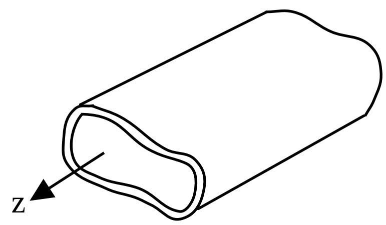
W.C.Chew
ECE 350 Lecture Notes

22. Hollow Waveguide.



A **hollow** cylindrical waveguide of uniform and arbitrary cross-section can guide waves. The fields inside a hollow waveguide can guide waves of both TE and TM types. When the field is of TE type, the electric field is purely transverse to the direction of wave propagation z; Hence $E_z = 0$. For TM fields, the magnetic field is purely transverse to the z-axis and hence, $H_z = 0$. Therefore, the field components of **TE fields** are

$$E_x, E_y, H_x, H_y, H_z,$$

and for **TM fields**, they are

$$H_x, H_y, E_x, E_y, E_z.$$

We can hence characterize **TE fields** as having $E_z = 0, H_z \neq 0$, and **TM fields** as $H_z = 0, E_z \neq 0$. Hence, the z-component of the **H** field can be used to characterize TE fields, while the z-component of the **E** field can be used to characterize TM fields in a hollow waveguide. Given E_z, and H_z, it will be desirable to derive the transverse components of the fields. We shall denote a vector transverse to $\hat{z}$ by a subscript s. In this notation, Maxwell's equations become

$$\left(\nabla_s + \hat{z}\frac{\partial}{\partial z}\right) \times (\mathbf{H}_s + \hat{z}H_z) = j\omega\epsilon(\mathbf{E}_s + \hat{z}E_z), \tag{1}$$

$$\left(\nabla_s + \hat{z}\frac{\partial}{\partial z}\right) \times (\mathbf{E}_s + \hat{z}E_z) = -j\omega\mu(\mathbf{H}_s + \hat{z}H_z), \tag{2}$$

where $\nabla_s = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}$, and $\mathbf{E}_s$ and $\mathbf{H}_s$ are the electric field and the magnetic field, respectively, transverse to the z directon. Equating the transverse components in (1) and (2), we have

$$\nabla_s \times \hat{z}H_z + \frac{\partial}{\partial z}\hat{z} \times \mathbf{H}_s = j\omega\epsilon\mathbf{E}_s, \tag{3}$$

$$\nabla_s \times \hat{z}E_z + \frac{\partial}{\partial z}\hat{z} \times \mathbf{E}_s = -j\omega\mu\mathbf{H}_s. \tag{4}$$

Substituting (4) for $\mathbf{H}_s$ into (3), we have

$$\nabla_s \times \hat{z}H_z + \frac{\partial}{\partial z}\hat{z} \times \frac{j}{\omega\mu}\left(\nabla_s \times \hat{z}E_z + \frac{\partial}{\partial z}\hat{z} \times \mathbf{E}_s\right) = j\omega\epsilon\mathbf{E}_s. \tag{5}$$

Using the vector identity

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}), \tag{6}$$

we can show that

$$\hat{z} \times \nabla_s \times \hat{z}E_z = \nabla_s(\hat{z} \cdot \hat{z}E_z) - \hat{z}E_z(\hat{z} \cdot \nabla_s) = \nabla_s E_z, \tag{7}$$

and

$$\hat{z} \times (\hat{z} \times \mathbf{E}_s) = \hat{z}(\hat{z} \cdot \mathbf{E}_s) - \mathbf{E}_s(\hat{z} \cdot \hat{z}) = -\mathbf{E}_s. \tag{8}$$

Hence, (5) becomes

$$\nabla_s \times \hat{z}H_z + \frac{j}{\omega\mu}\frac{\partial}{\partial z}\nabla_s E_z - \frac{j}{\omega\mu}\frac{\partial^2}{\partial z^2}\mathbf{E}_s = j\omega\epsilon\mathbf{E}_s. \tag{9}$$

If $\mathbf{E}$ is of the form $\mathbf{A}e^{-j\beta_z z} + \mathbf{B}e^{j\beta_z z}$, then $\frac{\partial^2}{\partial z^2} = -\beta_z^2$ and (9) becomes

$$\mathbf{E}_s = \frac{1}{\omega^2\mu\epsilon - \beta_z^2}\left[\frac{\partial}{\partial z}\nabla_s E_z - j\omega\mu\nabla_s \times \hat{z}H_z\right]. \tag{10}$$

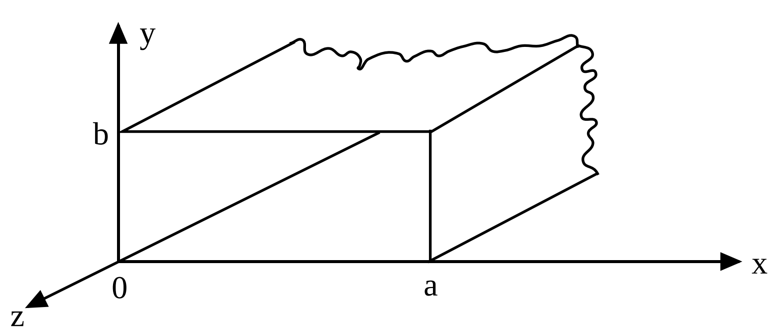
In a similar fashion, we obtain

$$\mathbf{H}_s = \frac{1}{\omega^2\mu\epsilon - \beta_z^2}\left[\frac{\partial}{\partial z}\nabla_s H_z + j\omega\epsilon\nabla_s \times \hat{z}E_z\right]. \tag{11}$$

The above equations can be used to derive the transverse components of the fields given the $\hat{z}$-components. Hence, in general, we only need to know the $\hat{z}$-components of the fields.

I. Rectangular Waveguides

Rectangular waveguides are a special case of cylindrical waveguides with uniform rectangular cross section. Hence, we can divide the waves inside the waveguide into TM and TE types.



TM Case, $H_z = 0, E_z \neq 0$

Inside the waveguide, we have a source free region, therefore

$$[\nabla^2 + \omega^2 \mu\epsilon]\mathbf{E} = 0, \tag{12}$$

or

$$[\nabla^2 + \omega^2 \mu\epsilon]E_z = 0. \tag{13}$$

Equation (13) admits solutions of the form

$$E_z = E_0 \left\{ \begin{array}{c} \sin\beta_x x \\ \cos\beta_x x \end{array} \right\} \left\{ \begin{array}{c} \sin\beta_y y \\ \cos\beta_y y \end{array} \right\} e^{-j\beta_z z}, \tag{14}$$

since

$$\frac{\partial^2}{\partial x^2} \left\{ \begin{array}{c} \sin\beta_x x \\ \cos\beta_x x \end{array} \right\} = \beta_x^2 \left\{ \begin{array}{c} \sin\beta_x x \\ \cos\beta_x x \end{array} \right\}, \tag{15}$$

$$\frac{\partial^2}{\partial y^2} \left\{ \begin{array}{c} \sin\beta_y y \\ \cos\beta_y y \end{array} \right\} = -\beta_y^2 \left\{ \begin{array}{c} \sin\beta_y y \\ \cos\beta_y y \end{array} \right\}, \quad \frac{\partial^2}{\partial z^2} e^{-j\beta_z z} = -\beta_z^2 e^{j\beta_z z}. \tag{16}$$

Therefore

$$(\nabla^2 + \omega^2\mu\epsilon)E_z = (-\beta_x^2 - \beta_y^2 - \beta_z^2 + \omega^2\mu\epsilon)E_z = 0. \tag{17}$$

This is only possible if

$$\beta_x^2 + \beta_y^2 + \beta_z^2 = \omega^2\mu\epsilon, \tag{18}$$

which is the **dispersion relation**. The boundary conditions require that

$$E_z(x = 0) = 0, \qquad E_z(y = 0) = 0. \tag{19}$$

Hence, the admissible solution is

$$E_z = E_0 \sin(\beta_x x)\sin(\beta_y y)e^{-j\beta_z z}. \tag{20}$$

Also, we require that

$$E_z(x = a) = 0, \qquad E_z(y = b) = 0. \tag{21}$$

This is only possible if $\sin(\beta_x a) = 0$ and $\sin(\beta_y b) = 0$, or

$$\beta_x a = m\pi, m = 0, 1, 2, \ldots, \qquad \beta_y b = n\pi, n = 0, 1, 2, 3, \ldots. \tag{22}$$

However, when m or $n = 0$, $E_z = 0$. Hence, we have

$$\beta_x = \frac{m\pi}{a}, \quad m \geq 1, \qquad \beta_y = \frac{n\pi}{b}, \quad n \geq 1, \tag{23}$$

which are the **guidance conditions**. To get the transverse **E** and **H** fields, we use (10) and (11)

$$E_x = \frac{1}{\omega^2\mu\epsilon - \beta_z^2}\frac{\partial}{\partial z}\frac{\partial}{\partial x}E_z = \frac{-j\beta_x\beta_z}{\beta_x^2+\beta_y^2}E_0\cos(\beta_x x)\sin(\beta_y y)e^{-j\beta_z z}, \tag{24}$$

$$E_y = \frac{1}{\omega^2\mu\epsilon - \beta_z^2}\frac{\partial}{\partial z}\frac{\partial}{\partial y}E_z = \frac{-j\beta_x\beta_z}{\beta_x^2+\beta_y^2}E_0\sin(\beta_x x)\cos(\beta_y y)e^{-j\beta_z z}, \tag{25}$$

$$H_x = \frac{j\omega\epsilon}{\omega^2\mu\epsilon - \beta_z^2}\frac{\partial}{\partial y}E_z = \frac{j\omega\epsilon\beta_y}{\beta_x^2+\beta_y^2}E_0\sin(\beta_x x)\cos(\beta_y y)e^{-j\beta_z z}, \tag{26}$$

$$H_y = \frac{-j\omega\epsilon}{\omega^2\mu\epsilon - \beta_z^2}\frac{\partial}{\partial x}E_z = \frac{-j\omega\epsilon\beta_x}{\beta_x^2+\beta_y^2}E_0\cos(\beta_x x)\sin(\beta_y y)e^{-j\beta_z z}. \tag{27}$$

We note that the electric fields satisfy their boundary conditions. From the dispersion relation (18), we have

$$\beta_z = \sqrt{\omega^2\mu\epsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}. \tag{28}$$

The solution that corresponds to a particular choice of m and n in (23) is known as the $\textbf{TM}_{mn}$ **mode**. For a given TM_{mn} mode, β_z will be pure imaginary if

$$\omega^2\mu\epsilon < \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2, \tag{29}$$

or

$$\omega < \frac{1}{\sqrt{\mu\epsilon}}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^{\frac{1}{2}}. \tag{30}$$

In this case, the mode is cutoff, and the fields decay in the $\hat{z}$-direction and become purely **evanescent**. We define the cutoff frequency for the TM_{mn} mode to be

$$\omega_{mnc} = \frac{1}{\sqrt{\mu\epsilon}}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^{\frac{1}{2}} = v\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^{\frac{1}{2}}. \tag{31}$$

The TM_{mn} mode will not propagate if

$$\omega < \omega_{mnc} \text{ or } f < f_{mnc}, \tag{32}$$

where $f_{mnc} = \frac{\omega_{mnc}}{2\pi}$, $f = \frac{\omega}{2\pi}$. The corresponding cutoff wavelength is

$$\lambda_{mnc} = 2\pi\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^{-\frac{1}{2}}. \tag{31a}$$

Only when the wavelength λ is smaller than this "size" can the wave "enter" the waveguide and be guided as the TM_{mn} mode.

To find the power flowing in the waveguide, we use the Poynting theorem.

$$S_z = E_x H_y^* - E_y H_x^*, \tag{33}$$

$$= \frac{\omega\epsilon\beta_x^2\beta_z}{(\beta_x^2+\beta_y^2)^2} |E_0|^2 \cos^2(\beta_x x)\sin^2(\beta_y y) + \frac{\omega\epsilon\beta_y^2\beta_z}{(\beta_x^2+\beta_y^2)^2} |E_0|^2 \sin^2(\beta_x x)\cos^2(\beta_y y)$$

$$= \frac{\omega\epsilon\beta_z}{(\beta_x^2+\beta_y^2)^2} |E_0|^2 [\beta_x^2 \cos^2(\beta_x x)\sin^2(\beta_y y) + \beta_y^2 \sin^2(\beta_x x)\cos^2(\beta_y y)]. \tag{34}$$

The total power

$$P_z = \int_0^b dy \int_0^a dx S_z = \frac{\omega\epsilon\beta_z ab |E_0|^2}{4(\beta_x^2+\beta_y^2)^2}(\beta_x^2+\beta_y^2) = \frac{\omega\epsilon\beta_z ab |E_0|^2}{4(\beta_x^2+\beta_y^2)}. \tag{35}$$

When $f < f_{mnc}$, β_z is purely imaginary and the power becomes purely reactive. No real power or time average power flows down a waveguide when all the modes are cutoff.

TE Case, $E_z = 0, H_z \neq 0$.

In this case,

$$H_z = H_0 \cos(\beta_x x)\cos(\beta_y y)e^{-j\beta_z z}, \tag{36}$$

so that from equations (10) and (11), we have,

$$E_x = -\frac{j\omega\mu}{\omega^2\mu\epsilon - \beta_z^2}\frac{\partial}{\partial y}H_z = \frac{j\omega\mu\beta_y}{\beta_x^2+\beta_y^2}H_0\cos(\beta_x x)\sin(\beta_y y)e^{-j\beta_z z}, \tag{37}$$

$$E_y = \frac{j\omega\mu}{\omega^2\mu\epsilon - \beta_z^2}\frac{\partial}{\partial x}H_z = \frac{-j\omega\mu\beta_x}{\beta_x^2+\beta_y^2}H_0\sin(\beta_x x)\cos(\beta_y y)e^{-j\beta_z z}, \tag{38}$$

$$H_x = \frac{1}{\omega^2\mu\epsilon - \beta_z^2}\frac{\partial}{\partial z}\frac{\partial}{\partial x}H_z = \frac{j\beta_x\beta_z}{\beta_x^2+\beta_y^2}H_0\sin(\beta_x x)\cos(\beta_y y)e^{-j\beta_z z}, \tag{39}$$

$$H_y = \frac{1}{\omega^2\mu\epsilon - \beta_z^2}\frac{\partial}{\partial z}\frac{\partial}{\partial y}H_z = \frac{j\beta_y\beta_z}{\beta_x^2+\beta_y^2}H_0\cos(\beta_x x)\sin(\beta_y y)e^{-j\beta_z z}, \tag{40}$$

where $\beta_x^2 + \beta_y^2 + \beta_z^2 = \beta^2 = \omega^2\mu\epsilon$. Matching boundary conditions for the tangential electric field requires that

$$\beta_x = \frac{m\pi}{a}, m = 0,1,2,3,\ldots, \quad \beta_y = \frac{n\pi}{b}, n = 0,1,2,3,\ldots. \tag{41}$$

Unlike the TM case, the TE case can have either m or n equal to zero. Hence, TE_{m0} or TE_{0n} modes exist. However, when both m and n are zero, $H_z = H_0 e^{-j\beta_z z}$, $H_x = H_y = 0$, and $\nabla \cdot \mathbf{H} \neq 0$, therefore, TE_{00} mode cannot exist.

For the TE_{mn} modes, the subscript m is associated with the longer side of the rectangular waveguide, while n is associated with the shorter side. In the case of TE_{m0} mode, $\beta_y = 0$, implying that $E_x = 0$, $E_y \neq 0$, $H_y = 0$, $H_x \neq 0$, $H_z \neq 0$. The fields resemble that of the TE_m mode in a **parallel plate waveguide**. For the general TE_{mn} mode, the dispersion relation is

$$\beta_z = \sqrt{\omega^2 \mu \epsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}. \tag{42}$$

Hence, the TE_{mn} mode and the TM_{mn} mode have the same cutoff frequency and they are **degenerate**.

Example: Designing a Waveguide to Propagate only the TE_{10} mode

The cutoff frequency of a TM_{mn} or a TE_{mn} mode is given by

$$\omega_{mnc} = \frac{1}{\sqrt{\mu\epsilon}} \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^{\frac{1}{2}}. \tag{43}$$

Usually, a is assumed to be larger than b so that TE_{10} mode has the lowest cutoff frequency, which is given by

$$f_{10c} = \frac{v}{2a} \text{ or } \lambda_{10c} = 2a, \tag{44}$$

where $v = \frac{1}{\sqrt{\mu\epsilon}}$, and $f_{10c} = \frac{\omega_{10c}}{2\pi}$. The next higher cutoff frequency is either f_{20c} or f_{01c} depending on the ratio of a to b.

$$f_{20c} = \frac{v}{a}, \quad f_{01c} = \frac{v}{2b}. \tag{45}$$

If $a > 2b$, $f_{20c} < f_{01c}$, and if $a < 2b$, $f_{20c} > f_{01c}$. $f_{20c} = f_{01c}$ if $a = 2b$. When $a = 2b$, and we want a waveguide to carry only the TE_{10} mode between 10 GHz and 20 GHz. Therefore, we want $f_{10c} = 10$ GHz, and $f_{20c} = f_{01c} =$ 20GHz. If the waveguide is filled with air, then $v = 3 \times 10^8 \frac{m}{s}$, and we deduce that

$$a = \frac{v}{2f_{10c}} = 1.5\text{cm}, \quad b = \frac{v}{2f_{01c}} = 0.75. \tag{46}$$

In such a rectangular waveguide, only the TE_{10} will propagate above 10 GHz and below 20 GHz. The other modes are all cutoff. Note that no mode could propagate below 10 GHz.