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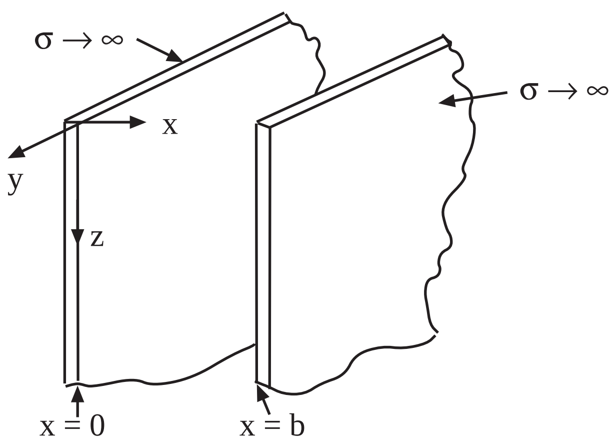
ECE 350 Lecture Notes

21. Infinite Parallel Plate Waveguide.



We have studied TEM (transverse electromagnetic) waves between two pieces of parallel conductors in the transmission line theory. We shall study other kinds of waves between two infinite parallel plates, or planes. We have learnt earlier that for a plane wave incident on a plane interface, the wave can be categorized into TE (transverse electric) with electric field polarized in the y-direction. Hence, between a parallel plate waveguide, we shall look for solutions of TE type with $\mathbf{E} = \hat{y}E_y$, or TM (transverse magnetic) type with $\mathbf{H} = \hat{y}H_y$. We shall assume that the field does not vary in the y-direction so that $\frac{\partial}{\partial y} = 0$.

We have shown earlier that if $\nabla \cdot \mathbf{E} = 0$, the equation for the $\mathbf{E}$ field in a source region is

$$(\nabla^2 + \omega^2\mu\epsilon)\mathbf{E} = 0. \tag{1}$$

If $\nabla \cdot \mathbf{H} = 0$, the equation for the $\mathbf{H}$ field is

$$(\nabla^2 + \omega^2\mu\epsilon)\mathbf{H} = 0. \tag{2}$$

Since $\frac{\partial}{\partial y} = 0$, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$ in these two equations.

I. TM Case, $\mathbf{H} = \hat{y}H_y$.

In this case,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \omega^2\mu\epsilon\right) H_y = 0. \tag{3}$$

If we assume that

$$H_y = A(x)e^{-j\beta_z z}, \tag{4}$$

substituting (4) into (3), we have

$$\left[\frac{d^2}{dx^2} + \omega^2\mu\epsilon - \beta_z^2\right] A(x) = 0. \tag{5}$$

Letting $\beta_x^2 = \omega^2\mu\epsilon - \beta_z^2$, (5) becomes

$$\left[\frac{d^2}{dx^2} + \beta_x^2\right] A(x) = 0, \tag{6}$$

where the independent solutions are

$$A(x) = \left\{ \begin{array}{c} \cos\beta_x x \\ \sin\beta_x x \end{array} \right. . \tag{7}$$

Hence, H_y is of the form

$$H_y = H_0 \left\{ \begin{array}{c} \cos\beta_x x \\ \sin\beta_x x \end{array} \right\} e^{-j\beta_z z}, \tag{8}$$

where

$$\beta_x^2 + \beta_z^2 = \omega^2\mu\epsilon = \beta^2, \tag{9}$$

which are the **dispersion relation** for plane waves. We can also define $\beta_x = \beta\cos\theta$, $\beta_z = \beta\sin\theta$ so that (9) is automatically satisfied.

To decide a viable solution from (8), we look at the boundary conditions for the **E**-field at the metallic plates. From $\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E}$, we have

$$j\omega\epsilon E_x = \frac{\partial}{\partial y}H_z - \frac{\partial}{\partial z}H_y, \tag{10}$$

(where $\frac{\partial}{\partial y}H_z = 0$ in the above equation) or

$$E_x = \frac{\beta_z}{\omega\epsilon}H_0 \left\{ \begin{array}{c} \cos\beta_x x \\ \sin\beta_x x \end{array} \right\} e^{-j\beta_z z}, \tag{11}$$

and

$$j\omega\epsilon E_z = \frac{\partial}{\partial x}H_y - \frac{\partial}{\partial y}H_x, \tag{12}$$

(where $\frac{\partial}{\partial y}H_x = 0$ in the above equation) or

$$E_z = -\frac{\beta_x}{j\omega\epsilon}H_0 \left\{ \begin{array}{c} \sin\beta_x x \\ -\cos\beta_x x \end{array} \right\} e^{-j\beta_z z}. \tag{13}$$

The boundary conditions require that $E_z(x=0) = E_z(x=b) = 0$. Only the first solution gives $E_z(x=0) = 0$. Hence, we eliminate the second solution, or

$$E_z = -\frac{\beta_x}{j\omega\epsilon} H_0 \sin(\beta_x x) e^{-j\beta_z z}. \tag{14}$$

In order for $E_z(x=b) = 0$, we require that

$$\sin \beta_x b = 0, \tag{15}$$

or

$$\beta_x b = m\pi, \qquad m = 0, \pm 1, \pm 2, \pm 3, \ldots, \tag{16}$$

and consequently,

$$\beta_x = \frac{m\pi}{b}, \qquad m = 0, \pm 1, \pm 2, \pm 3, \ldots. \tag{17}$$

This is known as the **guidance condition** for the waveguide. Finally, we have

$$H_y = H_0 \cos\left(\frac{m\pi}{b} x\right) e^{-j\beta_z z}, \tag{18}$$

$$E_x = \frac{\beta_z}{\omega\epsilon} H_0 \cos\left(\frac{m\pi x}{b}\right) e^{-j\beta_z z}, \tag{19}$$

$$E_z = -\frac{m\pi}{j\omega\epsilon b} H_0 \sin\left(\frac{m\pi x}{b}\right) e^{-j\beta_z z}, \tag{20}$$

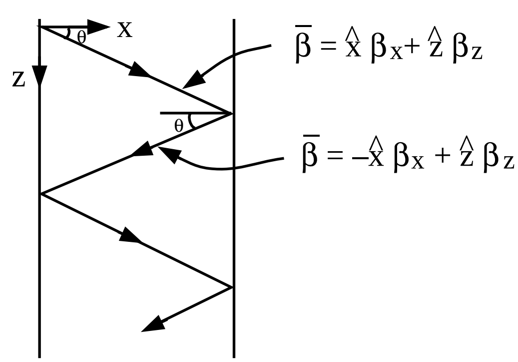
where

$$\beta_z = \left[\omega^2 \mu\epsilon - \left(\frac{m\pi}{b}\right)^2\right]^{\frac{1}{2}}, \tag{21}$$

which is the **dispersion relation** for the parallel plate waveguide. Equation (18) can be written as

$$H_y = \frac{H_0}{2}[e^{j\beta_x x} + e^{-j\beta_x x}]e^{-j\beta_z z} = \frac{H_0}{2}[e^{j\beta_x x - j\beta_z z} + e^{-j\beta_x x - j\beta_z z}]. \tag{22}$$

The first term in the above represents a plane wave propagating in the positive $\hat{z}$-direction and the negative $\hat{x}$-direction, while the second term corresponds to a wave propagating in the positive x and z directions. Hence, the field in between a parallel plate waveguide consists of a plane wave bouncing back and forth between the two plates, as shown.



Since we define $\beta_x = \beta \cos\theta$, $\beta_z = \beta \sin\theta$, the wave propagates in a direction making an angle θ with the $\hat{x}$-direction. Since the guidance condition requires that $\beta_x = \frac{m\pi}{b} = \beta \cos\theta$, the plane wave can be guided only for discrete values of θ.

From (21), we note that for different m's, β_z will assume different values. When $m = 0$, $\beta_z = \omega\sqrt{\mu\epsilon}$, $E_z = 0$, and we have a **TEM mode**. When $m > 0$, we have a **TM mode** of order m; we call it a TM_m mode. Hence, there are infinitely many solutions to Maxwell's equations between a parallel plate waveguide with the field given by (18), (19), (20), and the dispersion relation given by (21) where $m = 0, 1, 2, 3, \ldots$.

II. Cutoff Frequency

From (21), for a given TM_m mode, if $\omega\sqrt{\mu\epsilon} < \frac{m\pi}{b}$, then β_z is pure imaginary. In this case, the wave is purely decaying in the $\hat{z}$-direction, and it is **evanescent** and **non-propagating**. For a given TM_m mode, we can always lower the frequency so that this occurs. When this happens, we say that the mode is **cut off**. The **cutoff frequency** is the frequency for which a given TM_m mode becomes **cutoff** when the frequency of the TM_m mode is lower than this **cutoff frequency**. Hence,

$$\omega_{mc} = \frac{m\pi}{b\sqrt{\mu\epsilon}} \text{ or } f_{mc} = \frac{m}{2b\sqrt{\mu\epsilon}} = \frac{mv}{2b}. \tag{23}$$

When

$$\frac{(m+1)v}{2b} > f > \frac{mv}{2b} > \frac{(m-1)v}{2b} > \frac{(m-2)v}{2b} > \ldots > 0, \tag{24}$$

the TEM mode plus all the TM_n modes, where $0 < n \le m$ are **propagating** or **guided** while the TM_{m+1} and higher order modes are **evanescent** or **cutoff**. For the parallel plate waveguide, there is one mode with zero cutoff frequency and hence is guided for all frequencies. This is the **TEM mode** which is equivalent to the transmission line mode.

The wavelength that corresponds to the cutoff frequency is known as the **cutoff wavelength**, i.e.,

$$\lambda_{mc} = \frac{v}{f_{mc}} = \frac{2b}{m}. \tag{25}$$

When $\lambda < \lambda_{mc}$, the corresponding TM_m mode will be guided. You can think of λ as some kind of the "size" of the wave, and that only when the "size" of the wave is less than λ_{mc} can a wave "enter" the waveguide. Notice that λ_{mc} is proportional to the physical size of the waveguide.

IV. TE Case, $\mathbf{E} = \hat{y}E_y$.

The field for the TE case can be derived similarly to the TM case. The electric field is polarized in the $\hat{y}$-direction, and satisfies

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \omega^2\mu\epsilon\right] E_y = 0. \tag{26}$$

The fields can be shown in a similar fashion to be

$$E_y = E_0 \sin(\beta_x x) e^{-j\beta_z z}, \tag{27}$$

$$H_x = -\frac{\beta_z}{\omega\mu} E_0 \sin(\beta_x x) e^{-j\beta_z z}, \tag{28}$$

$$H_z = -\frac{\beta_x}{j\omega\mu} E_0 \cos(\beta_x x) e^{-j\beta_z z}. \tag{29}$$

The boundary conditions are

$$E_y(x=0) = 0, \quad E_y(x=b) = 0. \tag{30}$$

This gives

$$\beta_x = \frac{m\pi}{b}, \tag{31}$$

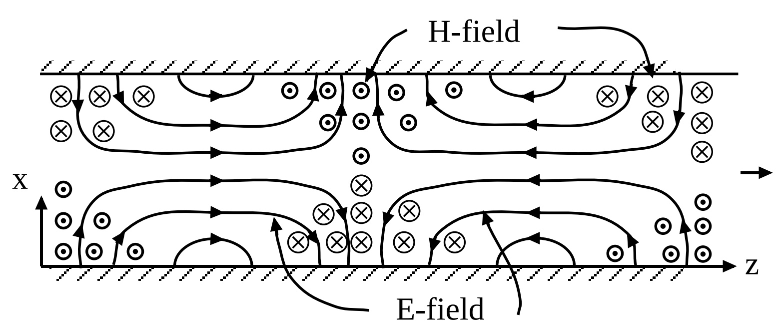
as before, where $\beta_x^2 + \beta_z^2 = \omega^2\mu\epsilon$. Hence, the TE_m modes have the same dispersion relation and cut-off frequency as the TM_m mode. However, when $m = 0$, $\beta_x = 0$, and (27)–(29) imply that we have zero field. Therefore, TE_0 mode does not exist. We say that TE_m and TM_m modes are **degenerate** when they have the same cutoff frequencies.

We can decompose (27) into plane waves, i.e.,

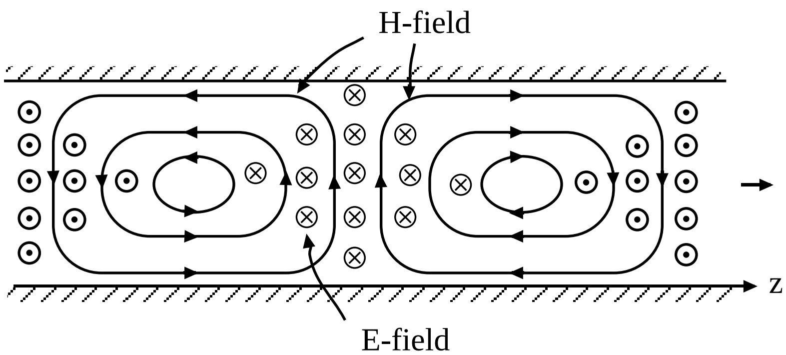
$$E_y = \frac{E_0}{2j}\left[e^{j\beta_x x - j\beta_z z} - e^{-j\beta_x x - j\beta_z z}\right], \tag{32}$$

and interpret the above as bouncing waves. Compared to (22), we see that the two bouncing waves in (32) are of the opposite signs whereas that in (22) are of the same sign. This is because the electric field has to vanish on the plates while the magnetic field need not.

TM_1 mode field



TE_1 mode field



The sketch of the fields for TM_1 and TE_1 modes are as shown above. For the TM mode, $H_z = 0$, and $E_z \neq 0$, while for the TE mode, $E_z = 0$, and $H_z \neq 0$. Tangential electric field is zero on the plates while tangential magnetic field is not zero on the plates. The above is the instantaneous field plots. $\mathbf{E} \times \mathbf{H}$ is in the direction of propagation of the waves.

III. Phase and Group Velocities.

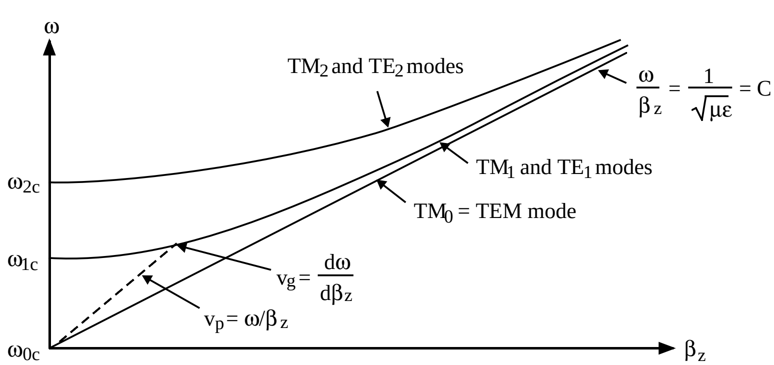
The phase velocity in the $\hat{z}$-direction of a wave in a waveguide is defined to be

$$v_p = \frac{\omega}{\beta_z} = \frac{\omega}{\left[\omega^2\mu\epsilon - \left(\frac{m\pi}{b}\right)^2\right]^{\frac{1}{2}}} = \frac{1}{\sqrt{\mu\epsilon}\left[1 - \left(\frac{f_{mc}}{f}\right)^2\right]^{\frac{1}{2}}}, \tag{33}$$

which is always **larger** than the speed of light for $f > f_{mc}$. The group velocity is

$$v_g = \frac{d\omega}{d\beta_z} = \left(\frac{d\beta_z}{d\omega}\right)^{-1} = \frac{\left[\omega^2\mu\epsilon - \left(\frac{m\pi}{b}\right)^2\right]^{\frac{1}{2}}}{\omega\mu\epsilon} = \frac{\left[1 - \left(\frac{f_{mc}}{f}\right)^2\right]^{\frac{1}{2}}}{\sqrt{\mu\epsilon}}, \tag{34}$$

which is always less than the speed of light.



Since $\beta_z = \frac{\omega}{c}\left[1 - \left(\frac{\omega_{mc}}{\omega}\right)^2\right]^{\frac{1}{2}}$, a plot of ω versus β_z is as shown. When $\beta_z \to 0$, the group velocity becomes zero while the phase velocity approaches infinity. When $\beta_z \to \infty$, or $\omega \to \infty$, the group and phase velocities both approach the velocity of light in free-space which is the TEM wave velocity.