
When $\nabla \cdot \mathbf{E} = 0$, the electric field satisfies the wave equation

$$\nabla^2 \mathbf{E} + \beta^2 \mathbf{E} = 0,$$  \hspace{1cm} (1)

where $\beta^2 = \omega^2 \mu \varepsilon$. We have learnt that one of the many possible solutions to the above equation is

$$\mathbf{E} = \hat{x} E_0 e^{-j\beta z}.$$  \hspace{1cm} (2)

The expression $e^{-j\beta z}$, when viewed in three dimensions, has constant phase planes or wave fronts which are orthogonal to the $z$-axis.

To denote a plane wave propagating in other directions, we write it as

$$\mathbf{E} = \hat{a} E_0 e^{-j\beta_0 x - j\beta_0 y - j\beta_0 z},$$  \hspace{1cm} (3)

where $\hat{a}$ is a constant unit vector, and $E_0$ a constant. If we substitute (3) into (1), we obtain

$$[-\beta_x^2 - \beta_y^2 - \beta_z^2 + \beta^2]E_0 = 0.$$  \hspace{1cm} (4)

In order for (3) to satisfy (1) and that $E_0 \neq 0$, we require that

$$\beta_x^2 + \beta_y^2 + \beta_z^2 = \beta^2 = \omega^2 \mu \varepsilon.$$  \hspace{1cm} (5)

If we define a vector $\mathbf{\beta} = \hat{x}\beta_x + \hat{y}\beta_y + \hat{z}\beta_z$, and $\mathbf{r} = \hat{x}x + \hat{y}y + \hat{z}z$, then (3) can be written as

$$\mathbf{E} = \hat{a} E_0 e^{-j\mathbf{\beta} \cdot \mathbf{r}},$$  \hspace{1cm} (6)

where the magnitude of $\mathbf{\beta}$ is

$$|\mathbf{\beta}| = [\beta_x^2 + \beta_y^2 + \beta_z^2]^{\frac{1}{2}} = \beta.$$  \hspace{1cm} (7)
Equation (6) is a concise way to write a solution to (1). Since $\nabla \cdot \mathbf{E} = 0$ using (3), we note that

$$
\nabla \cdot \mathbf{E} = -j [\hat{x} \beta_x + \hat{y} \beta_y + \hat{z} \beta_z] \cdot \hat{a} E_0 e^{-j \beta r}.
$$
(8)

Therefore, in order for $\nabla \cdot \mathbf{E} = 0$, we require that

$$
\beta \cdot \hat{a} = 0.
$$
(9)

To explore further how the function $e^{-j \beta r}$ look like, we assume $\beta$ to be pointing in a direction as shown in the figure. The value of $\beta \cdot \mathbf{r}$ is constant on a plane that is orthogonal to $\beta$.

That is

$$
\beta \cdot \mathbf{r} = |\beta| |\mathbf{r}| \cos \theta = \beta (OA),
$$
(10)

for all $\mathbf{r}$ on the plane $S$ that is orthogonal to $\beta$. Hence, $S$ is the constant phase plane of $e^{-j \beta r} = e^{-j \beta (OA)}$. As one moves progressively in the $\beta$ direction, the function $e^{-j \beta r}$ has a phase that is linearly decreasing with distance. Therefore, $e^{-j \beta r}$ denotes a plane wave that is propagating in the $\beta$ direction. When $\beta$ is pointing in the $z$-direction, such that $\beta = \hat{z} \beta$, then $e^{-j \beta r} = e^{-j \beta z}$, which is our familiar solution of a plane wave propagating in the $z$-direction.

An example of a plane wave electric field satisfying Maxwell’s equations is

$$
\mathbf{E} = \hat{y} E_0 e^{-j \beta_x x - j \beta_z z},
$$
(11)

where $\beta_x^2 + \beta_z^2 = \beta^2$. The corresponding magnetic field can be derived using Maxwell’s equations.

$$
\nabla \times \mathbf{E} = -j \omega \mu \mathbf{H}.
$$
(12)

Hence,

$$
\mathbf{H} = \frac{-1}{j \omega \mu} \left( \hat{z} \frac{\partial}{\partial x} E_y - \hat{x} \frac{\partial}{\partial z} E_y \right)
= (\hat{z} \beta_x - \hat{x} \beta_z) \frac{E_0}{\omega \mu} e^{-j \beta_x x - j \beta_z z}.
$$
(13)
In general, when $\nabla$ operates on a plane wave phasor described by $e^{-j\beta r}$, it transforms into $-j\beta$. This is obvious also from Equation (8). Therefore, from (12), we can express

$$H = \frac{1}{\omega \mu} \beta \times E.$$  \hspace{1cm} (14)

Therefore, $H$ is orthogonal to both $E$ and $\beta$, or that $H \cdot E = 0$, and that $H \cdot \beta = 0$, in addition to $E \cdot \beta = 0$. Furthermore, $E \times H$ points in the direction of $\beta$. Therefore, for a plane electromagnetic wave, $E$, $H$, and $\beta$ form a right-handed orthogonal system. It is also a transverse electromagnetic (TEM) wave.