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ECE 350 Lecture Notes

17. Complex Poynting Theorem.

The complex Poynting vector is defined to be

$$\underline{\mathbf{S}} = \underline{\mathbf{E}} \times \underline{\mathbf{H}}^*. \tag{1}$$

It has the dimension of watt$/m^2$ and it denotes the flow of complex power. (We have used underbars to denote complex vectors).

Before we proceed further, let us look at Maxwell's equations for the phasor field. In phasor representation, Maxwell's equations become

$$\nabla \times \underline{\mathbf{H}} = \underline{\mathbf{J}} + j\omega\epsilon\underline{\mathbf{E}}, \tag{2}$$
$$\nabla \times \underline{\mathbf{E}} = -j\omega\mu\underline{\mathbf{H}}. \tag{3}$$

First, we study the divergence property of (1),

$$\nabla \cdot (\underline{\mathbf{E}} \times \underline{\mathbf{H}}^*) = \underline{\mathbf{H}}^* \cdot \nabla \times \underline{\mathbf{E}} - \underline{\mathbf{E}} \cdot \nabla \times \underline{\mathbf{H}}^*. \tag{4}$$

Substituting (2) and (3) into (4), we have

$$\begin{aligned}\nabla \cdot (\underline{\mathbf{E}} \times \underline{\mathbf{H}}^*) &= -j\omega\mu\underline{\mathbf{H}} \cdot \underline{\mathbf{H}}^* + j\omega\epsilon\underline{\mathbf{E}} \cdot \underline{\mathbf{E}}^* - \underline{\mathbf{E}} \cdot \underline{\mathbf{J}}^* \\ &= -j\omega[\mu\,|\underline{\mathbf{H}}|^2 - \epsilon\,|\underline{\mathbf{E}}|^2] - \underline{\mathbf{E}} \cdot \underline{\mathbf{J}}^*. \end{aligned} \tag{5}$$

Comparing with (16.4), (5) involves the difference of the stored energy terms rather than the sum.

We have shown that for two quantities,

$$\mathbf{A}(z,t) = \Re e[\underline{\mathbf{A}}(z)e^{j\omega t}], \tag{6}$$
$$\mathbf{B}(z,t) = \Re e[\underline{\mathbf{B}}(z)e^{j\omega t}]. \tag{7}$$

The time average of $A(z,t)B(z,t)$, denoted by $\langle A, B\rangle$ is given by

$$\langle A, B\rangle = \frac{1}{2}\Re e[\underline{\mathbf{A}}(z)\underline{\mathbf{B}}^*(z)]. \tag{8}$$

Therefore,

$$\langle \mathbf{S}\rangle = \langle \mathbf{E} \times \mathbf{H}\rangle = \frac{1}{2}\Re e[\underline{\mathbf{E}} \times \underline{\mathbf{H}}^*] = \frac{1}{2}\Re e[\underline{\mathbf{S}}]. \tag{9}$$

The imaginary part of $\underline{\mathbf{S}}$ corresponds to instantaneous power that time averages to zero. It is also known as the reactive power. We can also convert (5) into integral from using the divergence theorem,

$$\oint_A (\underline{\mathbf{E}} \times \underline{\mathbf{H}}^*) \cdot \hat{n}\, dA = -j\omega \oint_V [\mu\, |\mathbf{H}|^2 - \epsilon\, |\mathbf{E}|^2]\, dV - \oint_V \sigma\, |\mathbf{E}|^2\, dV, \tag{10}$$

where we have assumed that $\mathbf{J} = \sigma\mathbf{E}$. If μ, ϵ, and σ are all real, then

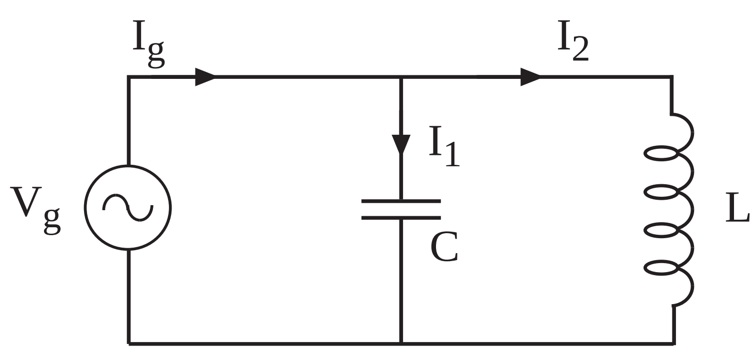
$$\oint_A \Re e(\mathbf{E} \times \mathbf{H}^*) \cdot \hat{n}\, dA = -\oint_V \sigma\, |\mathbf{E}|^2\, dV, \tag{11}$$

and

$$\oint_A \Im m(\mathbf{E} \times \mathbf{H}^*) \cdot \hat{n}\, dA = -\omega \oint_V [\mu\, |\mathbf{H}|^2 - \epsilon\, |\mathbf{E}|^2]\, dV. \tag{12}$$

We see that the real part of the power corresponds to power dissipated in V while the imaginary part of the power corresponds to difference in the magnetic energy stored and the electric energy stored. Hence, if a system has equal amount of magnetic and electric energy stored, it does not consume any reactive power.

Example of Reactive Power



We notice that in the complex Poynting theorem, the reactive power is proportional to $\omega(\mu|\mathbf{H}|^2 - \epsilon|\mathbf{E}|^2)$. It is zero when $\mu|\mathbf{H}|^2 = \epsilon|\mathbf{E}|^2$, or when the stored magnetic field energy equals the stored electric field energy. To comprehend this further, we look at a simple LC circuit driven by a time-harmonic voltage source.

At the resonant frequency of the tank circuit, $\omega = 1/\sqrt{LC}$, its input impedance is infinite, and hence $I_g = 0$. Therefore, there is no power delivered from the generator, be it real or reactive. However, $I_1 = -I_2 \neq 0$ at resonance, and as the tank circuit is resonating, the electric field energy stored in C is being converted into the magnetic field energy stored in L. Therefore,

$\frac{1}{2}L|I|^2 = \frac{1}{2}C|V|^2$ can be easily verified for a resonating tank circuit. This is precisely the case mentioned above.

Away from resonance,

$$I_g = V_g(j\omega C + \frac{1}{j\omega L}) = j\omega C V_g(1 - \frac{1}{\omega^2 LC}).$$

I_g is at 90° out-of-phase with V_g, and the complex power, $V_g I_g^*$ is purely imaginary. This implies that there is no time average power delivered by the source V_g, but it delivers nonzero reactive power. Away from resonance, the magnetic and electric stored energies are not in perfect balance with respect to each other, and we need to augment the system with external reactive power.