W.C.Chew
ECE 350 Lecture Notes

16. Real Poynting Theorem.

Since $\mathbf{E}\times\mathbf{H}$ has the dimension of watts/m^2, we can study its divergence property and its conservative property. Using the vector identity in (1.26), we have,

$$\nabla\cdot(\mathbf{E}\times\mathbf{H}) = \mathbf{H}\cdot\nabla\times\mathbf{E} - \mathbf{E}\cdot\nabla\times\mathbf{H}. \tag{1}$$

From Maxwell's equations, we can replace $\nabla\times\mathbf{E}$ by $-\frac{\partial\mathbf{B}}{\partial t}$ and $\nabla\times\mathbf{H}$ by $\frac{\partial\mathbf{D}}{\partial t}+\mathbf{J}$. Hence,

$$\begin{aligned}\nabla\cdot(\mathbf{E}\times\mathbf{H}) &= -\mathbf{H}\frac{\partial\mathbf{B}}{\partial t} - \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} - \mathbf{E}\cdot\mathbf{J}\\ &= -\mu\mathbf{H}\cdot\frac{\partial\mathbf{H}}{\partial t} - \epsilon\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t} - \mathbf{E}\cdot\mathbf{J}.\end{aligned} \tag{2}$$

We can show that

$$\frac{1}{2}\frac{\partial\left|\mathbf{H}\right|^2}{\partial t} = \mathbf{H}\cdot\frac{\partial\mathbf{H}}{\partial t}. \tag{3}$$

Hence,

$$\nabla\cdot(\mathbf{E}\times\mathbf{H}) = -\frac{\partial}{\partial t}\left(\frac{1}{2}\mu\left|\mathbf{H}\right|^2 + \frac{1}{2}\epsilon\left|\mathbf{E}\right|^2\right) - \mathbf{E}\cdot\mathbf{J}. \tag{4}$$

We can define

$$\mathbf{S} = \mathbf{E}\times\mathbf{H} \text{ Poynting vector (Power Flow Density } watt\, m^{-2}), \tag{5}$$

$$U_H = \frac{1}{2}\mu\left|\mathbf{H}\right|^2 \text{ Magnetic Energy Density } (joule\, m^{-3}), \tag{6}$$

$$U_E = \frac{1}{2}\epsilon\left|\mathbf{E}\right|^2 \text{ Electric Energy Density}(joule\, m^{-3}), \tag{7}$$

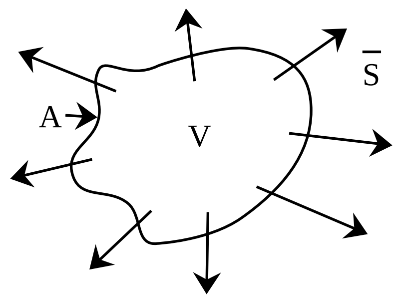
$$\mathbf{E}\cdot\mathbf{J} = \text{ Energy Dissipation Density}(watt\, m^{-3}). \tag{8}$$

U_H and U_E represent the energy stored in the magnetic field and electric field respectively. Alternatively, (4) becomes

$$\nabla\cdot\mathbf{S} = -\frac{\partial}{\partial t}(U_H + U_E) - \mathbf{E}\cdot\mathbf{J}. \tag{9}$$

Using the divergence theorem, (9) can be written in integral form,

$$\oint_A \mathbf{S} \cdot \hat{n}\, dA = -\frac{\partial}{\partial t} \int_V (U_H + U_E)\, dV - \int_V \mathbf{E} \cdot \mathbf{J}\, dV. \tag{10}$$



The equation says that the LHS will be positive only if there is a net outflow of the flux due to the vector field $\mathbf{S}$. If there is no current inside V so that $\mathbf{E} \cdot \mathbf{J} = 0$, then this is only possible if the stored energy $U_H + U_E$ inside V decreases with time.

If $\mathbf{J} = \sigma \mathbf{E}$, then the last term is $-\int \sigma |\mathbf{E}|^2\, dV$ is always negative. Hence, the last term tends to make $\oint_S \mathbf{S} \cdot \hat{n}\, dA$ negative, because energy dissipation has to be compensated by power flux flowing into V. The Poynting theorems (9) and (10) are statements of energy conservation. For example, for a plane wave,

$$\mathbf{E} = \hat{x} f(z - vt), \quad \mathbf{H} = \hat{y} \sqrt{\frac{\epsilon}{\mu}} f(z - vt), \tag{11}$$

then

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = \hat{z} \sqrt{\frac{\epsilon}{\mu}} f^2(z - vt). \tag{12}$$

Also,

$$U_E + U_H = \frac{1}{2} \epsilon f^2(z - vt) + \frac{1}{2} \epsilon f^2(z - vt) = \epsilon f^2(z - vt), \tag{13}$$

Therefore,

$$\mathbf{S} = \hat{z} \frac{1}{\sqrt{\mu \epsilon}} \epsilon f^2(z - vt) = \hat{z} v (U_E + U_H). \tag{14}$$

Hence, the velocity times the total energy density stored equals the power density flow in a plane wave.