W.C.Chew
ECE 350 Lecture Notes

14. Skin Depth and Plane Wave in a Lossy Medium.

We learn earlier that in a lossy medium, $\mathbf{J} = \sigma \mathbf{E}$, and from

$$\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J} = \epsilon \frac{\partial \mathbf{E}}{\partial t} + \sigma \mathbf{E}. \tag{1}$$

Using phasor technique, we can convert the above to

$$\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E} + \sigma\mathbf{E} = j\omega\underline{\epsilon}\mathbf{E}, \tag{2}$$

where

$$\underline{\epsilon} = \epsilon - j\frac{\sigma}{\omega}, \tag{3}$$

is the ***complex permittivity.*** Furthermore, using that

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}, \tag{4}$$

and that $\nabla \cdot \mathbf{H} = 0$, $\nabla \cdot \mathbf{E} = 0$, we can show that

$$\nabla^2 \mathbf{E} = -\omega^2 \mu \underline{\epsilon} \mathbf{E}, \tag{5}$$

$$\nabla^2 \mathbf{H} = -\omega^2 \mu \underline{\epsilon} \mathbf{H}. \tag{6}$$

[Refer to § 4 for details]. If we assume that $\mathbf{E} = \hat{x}E_x(z)$, then, we can show that

$$\frac{d^2}{dz^2}E_x(z) - \gamma^2 E_x(z) = 0, \tag{7}$$

where

$$\gamma = j\omega\sqrt{\mu\underline{\epsilon}} = \alpha + j\beta. \tag{7a}$$

The general solution to (7) is of the form

$$E_x(z) = c_1 e^{-\gamma z} + c_2 e^{\gamma z}. \tag{8}$$

If we assume that $c_2 = 0$, we have only

$$E_x(z) = c_1 e^{-\gamma z}. \tag{9}$$

We can convert the above into a real time quantity using phasor techniques, or

$$\begin{aligned} E_x(z,t) &= |c_1| \, \Re e[e^{-\alpha z - j\beta z + j\phi_1 + j\omega t}] \\ &= |c_1| \, e^{-\alpha z} \cos(\omega t - \beta z + \phi_1), \end{aligned} \tag{10}$$

where we have assumed that $c_1 = |c_1| e^{j\phi_1}$. Hence, we see that $E_x(z,t)$ is a wave that propagates to the right with velocity $v = \frac{\omega}{\beta}$ and attenuation constant α. We can find α from equation (7a), and

$$\gamma = \alpha + j\beta = j\omega\sqrt{\mu\left(\epsilon - j\frac{\sigma}{\omega}\right)} = j\omega\sqrt{\mu\epsilon\left(1 - j\frac{\sigma}{\omega\epsilon}\right)}. \tag{11}$$

The first term on the RHS of (1) is the displacement current term, while the second term is the conduction current term. From (2), we see that the ratio $\frac{\sigma}{\omega\epsilon}$ is the ratio of the conduction current to the displacement current in a lossy medium. $\frac{\sigma}{\omega\epsilon}$ is also known as the ***loss tangent*** of a lossy medium.

(i) When $\frac{\sigma}{\omega\epsilon} \ll 1$, the loss tangent is small, and the conduction current compared to the displacement current is small. The medium behaves more like a dielectric medium. In this case, we can use binomial expansions to approximate (11) to obtain

$$\gamma = j\omega\sqrt{\mu\epsilon}\left(1 - j\frac{1}{2}\frac{\sigma}{\omega\epsilon}\right) = \frac{1}{2}\sigma\sqrt{\frac{\mu}{\epsilon}} + j\omega\sqrt{\mu\epsilon}, \tag{12}$$

where

$$\alpha = \frac{1}{2}\sigma\sqrt{\frac{\mu}{\epsilon}}, \beta = \omega\sqrt{\mu\epsilon}. \tag{13}$$

(ii) When $\frac{\sigma}{\omega\epsilon} \gg 1$, the loss tangent is large because there is more conduction current than displacement current in the medium. In this case, the medium is conductive. According to equation (11), when $\frac{\sigma}{\omega\epsilon} \gg 1$, we have

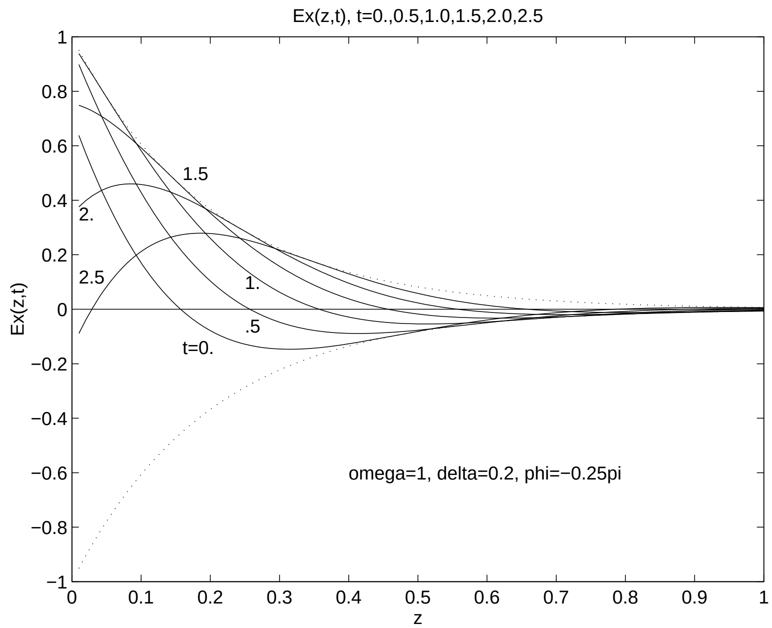
$$\gamma = j\omega\sqrt{-j\frac{\mu\sigma}{\omega}} = \sqrt{j\omega\mu\sigma} = (1+j)\sqrt{\frac{\omega\mu\sigma}{2}}. \tag{14}$$

Hence

$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \frac{1}{\delta}. \tag{15}$$

If we substitute $\alpha = \beta = \frac{1}{\delta}$ into (10), we have

$$E_x(z,t) = |c_1| e^{\frac{-z}{\delta}} \cos\left(\omega t - \frac{z}{\delta} + \phi_1\right). \tag{16}$$



This signal attenuates to e^{-1} of its original strength at $z = \delta$. Hence δ is also known as the ***penetration depth*** or the ***skin depth*** of a conductive medium. For other media, the penetration is $\frac{1}{\alpha}$, but for a conductive medium, it is

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}} = \sqrt{\frac{1}{\pi f \mu\sigma}}. \tag{17}$$

This skin depth decreases with increasing frequencies and increasing conductivities.

(iii) When $\frac{\sigma}{\omega\epsilon} \approx 1$, it is a general lossy medium, and we have to resort to complex arithmatics to find α and β.
If we square (11), we have

$$\alpha^2 - \beta^2 + 2j\alpha\beta = -\omega^2\mu(\epsilon - j\frac{\sigma}{\omega}), \tag{18}$$

or

$$\alpha^2 - \beta^2 = -\omega^2\mu\epsilon, \tag{19a}$$

$$2\alpha\beta = \omega\mu\sigma. \tag{19b}$$

Squaring (19a) and adding the square of (19b) to it, we have

$$(\alpha^2 - \beta^2)^2 + (2\alpha\beta)^2 = (\alpha^2 + \beta^2)^2 = \omega^4\mu^2\epsilon^2 + \omega^2\mu^2\sigma^2, \tag{20}$$

or

$$\alpha^2 + \beta^2 = \omega\mu\sqrt{\omega^2\epsilon^2 + \sigma^2}. \tag{21}$$

Combining with (19a), we deduce that

$$\alpha^2 = \frac{1}{2}(\omega\mu\sqrt{\omega^2\epsilon^2 + \sigma^2} - \omega^2\mu\epsilon), \tag{22a}$$

$$\beta^2 = \frac{1}{2}(\omega\mu\sqrt{\omega^2\epsilon^2 + \sigma^2} + \omega^2\mu\epsilon), \tag{22b}$$

Notice that when $\sigma = 0$, $\alpha = 0$.