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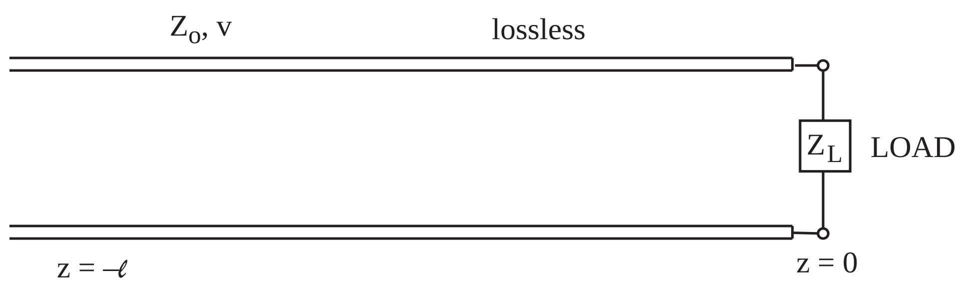
ECE 350 Lecture Notes

6. Terminated Uniform Lossless Transmission Lines



Consider a lossless transmission line terminated in a load of impedance Z_L. A wave traveling to the right will be reflected at the termination. In general, there will be both positive going and negative going waves on the line. Hence,

$$\tilde{V}(z) = V_0 e^{-j\beta z} + V_1 e^{+j\beta z}. \tag{1}$$

Here, $\gamma = j\beta$, $\alpha = 0$, because of no loss. The corresponding current, as in (5.32), is

$$\tilde{I}(z) = \frac{V_0}{Z_0} e^{-j\beta z} - \frac{V_1}{Z_0} e^{+j\beta z}, \tag{2}$$

where $Z_0 = \sqrt{\frac{L}{C}}$ and $\beta = \omega\sqrt{LC}$ for a lossless line.

At $z = 0$,

$$\frac{\tilde{V}(z=0)}{\tilde{I}(z=0)} = Z_L = \frac{V_0 + V_1}{V_0 - V_1} Z_0. \tag{3}$$

We can solve for V_1 in terms of V_0, i.e.

$$V_1 = \frac{Z_L - Z_0}{Z_L + Z_0} V_0. \tag{4}$$

If we define

$$\rho_v = \frac{Z_L - Z_0}{Z_L + Z_0}, \tag{5}$$

then $V_1 = \rho_v V_0$, and Equation (1) becomes

$$\tilde{V}(z) = V_0 e^{-j\beta z} + \rho_v V_0 e^{+j\beta z}. \tag{6}$$

In the above, ρ_v is the ratio of the negative going voltage amplitude to the positive going voltage amplitude at $z = 0$, and it is known as the ***voltage reflection coefficient***.

The ***current reflection coefficient*** is defined as the ratio of the negative going current to the positive going current at $z = 0$, and it is

$$\rho_i = \frac{I_1}{I_0} = -\frac{V_1}{V_0} = -\rho_v. \tag{7}$$

The current can be written as

$$\tilde{I}(z) = \frac{V_0}{Z_0} e^{-j\beta z} - \rho_v \frac{V_0}{Z_0} e^{j\beta z}. \tag{8}$$

The voltage and current in (6) and (8) are not constants of position. We can define a ***generalized impedance*** at position z to be

$$Z(z) = \frac{\tilde{V}(z)}{\tilde{I}(z)} = Z_0 \frac{e^{-j\beta z} + \rho_v e^{+j\beta z}}{e^{-j\beta z} - \rho_v e^{+j\beta z}}. \tag{9}$$

At $z = -l$, this becomes

$$Z(-l) = Z_0 \frac{e^{j\beta l} + \rho_v e^{-j\beta l}}{e^{j\beta l} - \rho_v e^{-j\beta l}}. \tag{10}$$

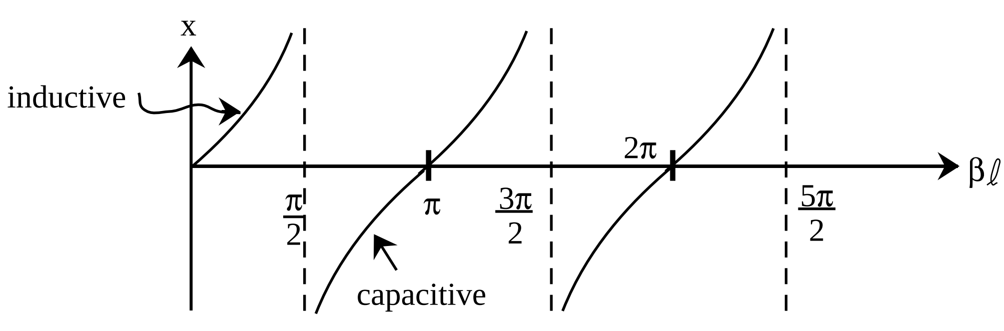
With ρ_v defined by (5), we can substitute it into (10) to give after some simplifications,

$$Z(-l) = Z_0 \frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l}. \tag{11}$$

Shorted Terminations

If Z_L is a short, or $Z_L = 0$, then,

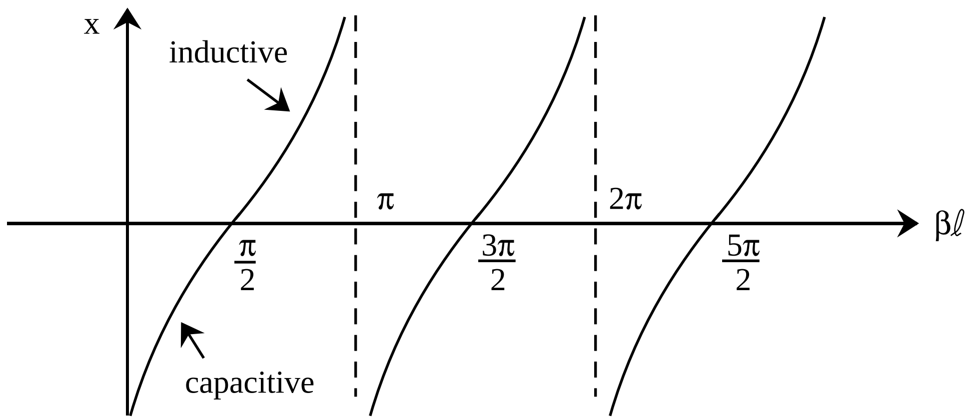
$$Z(-l) = jZ_0 \tan \beta l = jX. \tag{12}$$



Open-Circuit Terminations

If Z_L is an open circuit, $Z_L = \infty$, then

$$Z(-l) = -jZ_0 \cot \beta l = jX. \tag{13}$$



Standing Waves on a Lossless Transmission Line

The positive going wave in Equation (6) is

$$V_+(z) = V_0 e^{-j\beta z}, \tag{14}$$

and the negative going wave in Equation (6) is

$$V_-(z) = \rho_v V_0 e^{+j\beta z}. \tag{15}$$

We can define a ***generalized reflection coefficient*** to be the ratio of $V_+(z)$ to $V_-(z)$ at position z. Hence,

$$\Gamma(z) = \frac{V_-(z)}{V_+(z)} = \rho_v e^{2j\beta z}. \tag{16}$$

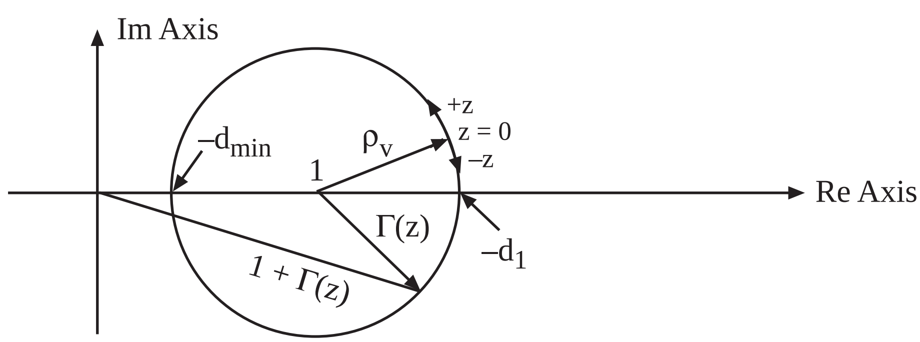
Hence,

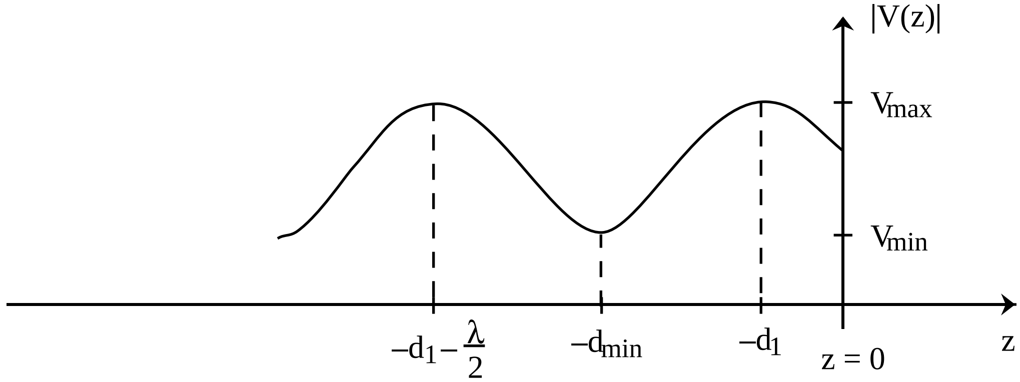
$$V(z) = V_0 e^{-j\beta z}[1 + \Gamma(z)]. \tag{17}$$

The magnitude of $V(z)$ is then

$$|V(z)| = |V_0|\,|1 + \Gamma(z)|\,. \tag{18}$$

A plot of $|V(z)|$ is as shown.





We can use the triangular inequality and show that

$$|V_0|\,(1-|\Gamma(z)|) \le |V(z)| \le |V_0|\,(1+|\Gamma(z)|). \tag{19}$$

From (16), $|\Gamma(z)| = |\rho_v|$, hence (19) becomes,

$$|V_0|\,(1-|\rho_v|) \le |V(z)| \le |V_0|\,(1+|\rho_v|). \tag{20}$$

The voltage standing wave ratio is defined to be V_{max}/V_{min}, and from (20), it is

$$\text{VSWR} = \frac{1+|\rho_v|}{1-|\rho_v|}. \tag{21}$$

If $\rho_v = 0$, then VSWR$=1$, and we have no reflected wave. We say that the load is matched to the transmission line. Note that $\rho_v = 0$ when $Z_L = Z_0$.

If $|\rho_v|$ =1, then VSWR$=\infty$, and we have a badly matched transmission line. In a passive load,

$$0 \le |\rho_v| \le 1. \tag{22}$$

$|\rho_v|$ =1 only when $Z_L = 0$, or $Z_L = \infty$ according to Equation (5). Hence,

$$1 \le \ \text{VSWR} \ < \infty. \tag{23}$$

VSWR is an indicator of how well a load is being matched to the transmission line. We can solve (21) for $|\rho_v|$ in terms of VSWR, i.e.

$$|\rho_v| = \frac{\text{VSWR}-1}{\text{VSWR}+1}. \tag{24}$$

Therefore, given the measurement of $VSWR$ on a terminated transmission line, we can deduce the magnitude of ρ_v. Furthermore, if we know the phase of ρ_v, we would be able to derive Z_L from (5), or

$$Z_L = Z_0\frac{1+\rho_v}{1-\rho_v}, \tag{25}$$

or

$$Z_L = Z_0\frac{1+|\rho_v|\,e^{j\theta_v}}{1-|\rho_v|\,e^{j\theta_v}}, \tag{26}$$

where

$$\rho_v = |\rho_v| \, e^{j\theta_v}. \tag{27}$$

Determining θ_v from $|V(z)|$

θ_v can be determined from the voltage standing wave measured. The voltage standing wave pattern is proportional to $|1 + \Gamma(z)|$, but $\Gamma(z)$ is related to ρ_v as

$$\Gamma(z) = \rho_v e^{2j\beta z}. \tag{28}$$

Writing the polar representation of ρ_v, we have,

$$\Gamma(z) = |\rho_v| \, e^{j(2\beta z + \theta_v)}. \tag{29}$$

However, we know that the first minimum value of $V(z)$ occurs when $\Gamma(z)$ is purely negative, or the phase of $\Gamma(z)$ is $-\pi$. This occurs at $z = -d_{min}$ first. In other words,

$$-2\beta d_{min} + \theta_v = -\pi. \tag{30}$$

Since d_{min} can be obtained from the voltage standing wave pattern measurement, and that $\beta = 2\pi/\lambda$, we deduce that

$$\theta_v = -\pi + \frac{4\pi}{\lambda} d_{min}. \tag{31}$$

Transmission Coefficients

It is sometimes useful to define a transmission coefficient on a transmission line. The transmission coefficient may be defined as the ratio of the voltage on the load to the amplitude of the incident voltage. Since

$$V(z) = V_0 e^{-j\beta z} + \rho_v V_0 e^{+j\beta z}. \tag{32}$$

The voltage at the load is $V(z = 0)$, and it is given by

$$V(0) = V_0(1 + \rho_v). \tag{33}$$

Since the amplitude of the incident voltage is V_0, we have

$$\tau_v = \frac{V(0)}{V_0} = 1 + \rho_v = \frac{2Z_L}{Z_L + Z_0}. \tag{34}$$