6. Terminated Uniform Lossless Transmission Lines

Consider a lossless transmission line terminated in a load of impedance $Z_L$. A wave traveling to the right will be reflected at the termination. In general, there will be both positive going and negative going waves on the line. Hence,

$$\tilde{V}(z) = V_0 e^{-j\beta z} + V_1 e^{+j\beta z}. \quad (1)$$

Here, $\gamma = j\beta$, $\alpha = 0$, because of no loss. The corresponding current, as in (5.32), is

$$\tilde{I}(z) = \frac{V_0}{Z_0} e^{-j\beta z} - \frac{V_1}{Z_0} e^{+j\beta z}, \quad (2)$$

where $Z_0 = \sqrt{LC}$ and $\beta = \omega \sqrt{LC}$ for a lossless line.

At $z = 0$,

$$\frac{\tilde{V}(z = 0)}{\tilde{I}(z = 0)} = Z_L = \frac{V_0 + V_1}{V_0 - V_1} \cdot Z_0. \quad (3)$$

We can solve for $V_1$ in terms of $V_0$, i.e.

$$V_1 = \frac{Z_L - Z_0}{Z_L + Z_0} V_0. \quad (4)$$

If we define

$$\rho_v = \frac{Z_L - Z_0}{Z_L + Z_0}, \quad (5)$$

then $V_1 = \rho_v V_0$, and Equation (1) becomes

$$\tilde{V}(z) = V_0 e^{-j\beta z} + \rho_v V_0 e^{+j\beta z}. \quad (6)$$

In the above, $\rho_v$ is the ratio of the negative going voltage amplitude to the positive going voltage amplitude at $z = 0$, and it is known as the voltage reflection coefficient.
The current reflection coefficient is defined as the ratio of the negative going current to the positive going current at \( z = 0 \), and it is

\[
\rho_i = \frac{I_1}{I_0} = -\frac{V_1}{V_0} = -\rho_v. \tag{7}
\]

The current can be written as

\[
\tilde{I}(z) = \frac{V_0}{Z_0} e^{-j\beta z} - \rho_v \frac{V_0}{Z_0} e^{j\beta z}. \tag{8}
\]

The voltage and current in (6) and (8) are not constants of position. We can define a generalized impedance at position \( z \) to be

\[
Z(z) = \frac{\tilde{V}(z)}{\tilde{I}(z)} = \frac{Z_0 e^{-j\beta z} + \rho_v e^{+j\beta z}}{e^{-j\beta z} - \rho_v e^{+j\beta z}}. \tag{9}
\]

At \( z = -l \), this becomes

\[
Z(-l) = Z_0 e^{j\beta l} + \rho_v e^{-j\beta l}. \tag{10}
\]

With \( \rho_v \) defined by (5), we can substitute it into (10) to give after some simplifications,

\[
Z(-l) = Z_0 \frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l}. \tag{11}
\]

**Shorted Terminations**

If \( Z_L \) is a short, or \( Z_L = 0 \), then,

\[
Z(-l) = jZ_0 \tan \beta l = jX. \tag{12}
\]

**Open-Circuit Terminations**

If \( Z_L \) is an open circuit, \( Z_L = \infty \), then

\[
Z(-l) = -jZ_0 \cot \beta l = jX. \tag{13}
\]
Standing Waves on a Lossless Transmission Line

The positive going wave in Equation (6) is

\[ V_+(z) = V_0 e^{-j\beta z}, \quad (14) \]

and the negative going wave in Equation (6) is

\[ V_-(z) = \rho_0 V_0 e^{+j\beta z}. \quad (15) \]

We can define a \textit{generalized reflection coefficient} to be the ratio of \( V_+(z) \) to \( V_-(z) \) at position \( z \). Hence,

\[ \Gamma(z) = \frac{V_-(z)}{V_+(z)} = \rho_0 e^{2j\beta z}. \quad (16) \]

Hence,

\[ V(z) = V_0 e^{-j\beta z}[1 + \Gamma(z)]. \quad (17) \]

The magnitude of \( V(z) \) is then

\[ |V(z)| = |V_0||1 + \Gamma(z)|. \quad (18) \]

A plot of \( |V(z)| \) is as shown.
We can use the triangular inequality and show that

$$|V_0| (1 - |\Gamma(z)|) \leq |V(z)| \leq |V_0| (1 + |\Gamma(z)|). \quad (19)$$

From (16), $|\Gamma(z)| = |\rho_v|$, hence (19) becomes,

$$|V_0| (1 - |\rho_v|) \leq |V(z)| \leq |V_0| (1 + |\rho_v|). \quad (20)$$

The voltage standing wave ratio is defined to be $V_{\text{max}}/V_{\text{min}}$, and from (20), it is

$$\text{VSWR} = \frac{1 + |\rho_v|}{1 - |\rho_v|}. \quad (21)$$

If $\rho_v = 0$, then $\text{VSWR} = 1$, and we have no reflected wave. We say that the load is matched to the transmission line. Note that $\rho_v = 0$ when $Z_L = Z_0$.

If $|\rho_v| = 1$, then $\text{VSWR} = \infty$, and we have a badly matched transmission line. In a passive load,

$$0 \leq |\rho_v| \leq 1. \quad (22)$$

$|\rho_v| = 1$ only when $Z_L = 0$, or $Z_L = \infty$ according to Equation (5). Hence,

$$1 \leq \text{VSWR} < \infty. \quad (23)$$

VSWR is an indicator of how well a load is being matched to the transmission line. We can solve (21) for $|\rho_v|$ in terms of VSWR, i.e.

$$|\rho_v| = \frac{\text{VSWR} - 1}{\text{VSWR} + 1}. \quad (24)$$

Therefore, given the measurement of VSWR on a terminated transmission line, we can deduce the magnitude of $\rho_v$. Furthermore, if we know the phase of $\rho_v$, we would be able to derive $Z_L$ from (5), or

$$Z_L = Z_0 \frac{1 + \rho_v}{1 - \rho_v}, \quad (25)$$

or

$$Z_L = Z_0 \frac{1 + |\rho_v| e^{j\theta_v}}{1 - |\rho_v| e^{j\theta_v}}. \quad (26)$$
where
\[ \rho_v = |\rho_v| e^{j\theta_v}. \]  

(27)

**Determining \( \theta_v \) from \(|V(z)|\)**

\( \theta_v \) can be determined from the voltage standing wave measured. The voltage standing wave pattern is proportional to \(|1 + \Gamma(z)|\), but \( \Gamma(z) \) is related to \( \rho_v \) as
\[ \Gamma(z) = \rho_v e^{2j\beta z}. \]  

(28)

Writing the polar representation of \( \rho_v \), we have,
\[ \Gamma(z) = |\rho_v| e^{j(2\beta z + \theta_v)}. \]  

(29)

However, we know that the first minimum value of \( V(z) \) occurs when \( \Gamma(z) \) is purely negative, or the phase of \( \Gamma(z) \) is \(-\pi\). This occurs at \( z = -d_{\text{min}} \) first. In other words,
\[ -2\beta d_{\text{min}} + \theta_v = -\pi. \]  

(30)

Since \( d_{\text{min}} \) can be obtained from the voltage standing wave pattern measurement, and that \( \beta = 2\pi/\lambda \), we deduce that
\[ \theta_v = -\pi + \frac{4\pi}{\lambda} d_{\text{min}}. \]  

(31)

**Transmission Coefficients**

It is sometimes useful to define a transmission coefficient on a transmission line. The transmission coefficient may be defined as the ratio of the voltage on the load to the amplitude of the incident voltage. Since
\[ V(z) = V_0 e^{-j\beta z} + \rho_v V_0 e^{+j\beta z}. \]  

(32)

The voltage at the load is \( V(z = 0) \), and it is given by
\[ V(0) = V_0(1 + \rho_v). \]  

(33)

Since the amplitude of the incident voltage is \( V_0 \), we have
\[ \tau_v = \frac{V(0)}{V_0} = 1 + \rho_v = \frac{2Z_L}{Z_L + Z_0}. \]  

(34)