4. Using Phasor Techniques to Solve Maxwell’s Equations

For a time-harmonic (simple harmonic) signal, Maxwell’s Equations can be easily solved using phasor techniques. For example, if we let

\[ \mathbf{H} = \Re \{ \hat{\mathbf{H}} e^{j\omega t} \}, \]

\[ \mathbf{E} = \Re \{ \hat{\mathbf{E}} e^{j\omega t} \}, \]

and substituting into (3.1), we have

\[ \Re \{ \nabla \times \hat{\mathbf{H}} e^{j\omega t} \} = \Re \left[ \frac{\partial}{\partial t} \hat{\mathbf{E}} e^{j\omega t} \right]. \]

(3)

We could replace \( \frac{\partial}{\partial t} \) by \( j\omega \) since the signal is time harmonic. Furthermore, we can remove the \( \Re \) operator and obtain

\[ \nabla \times \hat{\mathbf{H}} e^{j\omega t} = j\omega \varepsilon \hat{\mathbf{E}} e^{j\omega t}, \]

(4)

where \( e^{j\omega t} \) cancels out on both sides.

Equation (4) implies Equation (3). Also, any time dependence cancels out in the problem. Hence,

\[ \nabla \times \hat{\mathbf{H}} = j\omega \varepsilon \hat{\mathbf{E}}. \]

(5)

Similarly,

\[ \nabla \times \hat{\mathbf{E}} = -j\omega \mu \hat{\mathbf{H}}, \]

\[ \nabla \cdot \mu \hat{\mathbf{H}} = 0, \]

\[ \nabla \cdot \varepsilon \hat{\mathbf{E}} = 0. \]

(6-8)

Taking the curl of (6) and substituting (5) into it, we have

\[ \nabla \times \nabla \times \hat{\mathbf{E}} = -j\omega \mu \nabla \times \hat{\mathbf{H}} = \omega^2 \mu \varepsilon \hat{\mathbf{E}}. \]

(9)

Again, making use of the identity \( \nabla \times \nabla \times \hat{\mathbf{E}} = \nabla (\nabla \cdot \hat{\mathbf{E}}) - \nabla^2 \hat{\mathbf{E}} \), and \( \nabla \cdot \hat{\mathbf{E}} = 0 \), we have

\[ \nabla^2 \hat{\mathbf{E}} = -\omega^2 \mu \varepsilon \hat{\mathbf{E}}. \]

(10)

Similarly,

\[ \nabla^2 \hat{\mathbf{H}} = -\omega^2 \mu \varepsilon \hat{\mathbf{H}}. \]

(11)

These are the Helmholtz’s wave equations.

**Lossy Medium (Conductive Medium)**
Phasor technique is particularly appropriate for solving Maxwell’s equations in a lossy medium. In a lossy medium, Equation (3.1) becomes

\[ \nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}, \]

where \( \mathbf{J} \) is the induced currents in the medium, and hence,

\[ \mathbf{J} = \sigma \mathbf{E}. \]

Applying phasor technique to (12), we have

\[ \nabla \times \mathbf{H} = j \omega \epsilon \mathbf{E} + \sigma \mathbf{E} \]

\[ = j \omega \left( \epsilon - j \frac{\sigma}{\omega} \right) \mathbf{E}. \]

We can define the quantity

\[ \tilde{\epsilon} = \epsilon - j \frac{\sigma}{\omega} \]

to be the complex permittivity of the medium, and (14) becomes

\[ \nabla \times \mathbf{H} = j \omega \tilde{\epsilon} \mathbf{E}. \]

Notice that the only difference between (16) and (5) is the complex permittivity versus the real permittivity. If one goes about deriving the Helmholtz wave equations for a lossy medium, the results are

\[ \nabla^2 \tilde{\mathbf{E}} = -j \omega \mu \tilde{\epsilon} \tilde{\mathbf{E}}, \]

\[ \nabla^2 \tilde{\mathbf{H}} = -j \omega \mu \tilde{\epsilon} \tilde{\mathbf{H}}. \]

Hence, a lossy medium is easily treated using phasor technique by replacing a real permittivity with a complex permittivity.

If we restrict ourselves to one dimension, Equation (17), for instance, becomes of the form

\[ \frac{d^2}{dz^2} \tilde{E}_x(z) - \gamma^2 \tilde{E}_x(z) = 0, \]

where

\[ \gamma = j \omega \sqrt{\mu \epsilon} = j \omega \sqrt{\mu \left( \epsilon - j \frac{\sigma}{\omega} \right)} = \alpha + j \beta. \]

The general solution to (19) is of the form

\[ \tilde{E}_x(z) = C_1 e^{-\gamma z} + C_2 e^{+\gamma z}. \]

In real space time,

\[ E_x(z, t) = \Re \left[ \tilde{E}_x(z) e^{j \omega t} \right] \]

\[ = \Re \left[ C_1 e^{-\gamma z} e^{j \omega t} \right] + \Re \left[ C_2 e^{+\gamma z} e^{j \omega t} \right] \]
If \( C_1 = |C_1| e^{j\phi_1}, \quad C_2 = |C_2| e^{j\phi_2}, \quad \gamma = \alpha + j\beta, \) then

\[
E_x(z, t) = |C_1| \cos(\omega t - \beta z + \phi_1)e^{-\alpha z} + |C_2| \cos(\omega t + \beta z + \phi_2)e^{\alpha z}. \tag{24}
\]

Note that one of the solutions in (24) is decaying with \( z \) while another solution is growing with \( z \). The function \( \cos(\omega t \pm \beta z + \phi) \) can be written as \( \cos[\pm\beta(z \pm \frac{\omega}{\beta} t) + \phi] \). Hence, it moves with a velocity

\[
v = \frac{\omega}{\beta}. \tag{25}\]

Depending on its sign, it moves either in the positive or negative \( z \) direction. In the above, \( \gamma \) is the propagation constant, \( \alpha \) is the attenuation constant while \( \beta \) is the phase constant.

**Intrinsic Impedance**

The intrinsic impedance can be easily derived also in the phasor world. The phasor representation of Equation (3.23) is

\[
\frac{d}{dz} \bar{E}_x = -j \omega \mu \bar{H}_y. \tag{26}
\]

A corresponding one for \( \bar{H}_y \) is

\[
\frac{d}{dz} \bar{H}_y = -j \omega \epsilon \bar{E}_x. \tag{27}\]

If we now let \( \bar{E}_x = E_0 e^{-\gamma z}, \bar{H}_y = H_0 e^{-\gamma z} \), and using them in (26) yields

\[
-\gamma E_0 e^{-\gamma z} = -j \omega \mu H_0 e^{-\gamma z}. \tag{28}\]

The above implies that

\[
\eta = \frac{E_0}{H_0} = \frac{j \omega \mu}{\gamma} = \sqrt{\frac{\mu}{\epsilon}}. \tag{29}\]

For a lossy medium, we replace \( \epsilon \) by the complex permittivity and the intrinsic impedance becomes

\[
\eta = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu}{\epsilon - j \frac{\sigma}{\omega}}} = \sqrt{\frac{j \omega \mu}{\sigma + j \omega \mu}}. \tag{30}\]

The above is obviously a complex number.