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ECE 350 Lecture Notes

4. Using Phasor Techniques to Solve Maxwell's Equations

For a time-harmonic (simple harmonic) signal, Maxwell's Equations can be easily solved using phasor techniques. For example, if we let

$$\mathbf{H} = \Re e[\tilde{\mathbf{H}} e^{j\omega t}], \tag{1}$$

$$\mathbf{E} = \Re e[\tilde{\mathbf{E}} e^{j\omega t}], \tag{2}$$

and substituting into (3.1), we have

$$\Re e[\nabla \times \tilde{\mathbf{H}} e^{j\omega t}] = \Re e \left[\frac{\partial}{\partial t} \epsilon \tilde{\mathbf{E}} e^{j\omega t} \right]. \tag{3}$$

We could replace $\frac{\partial}{\partial t}$ by $j\omega$ since the signal is time harmonic. Furthermore, we can remove the $\Re e$ operator and obtain

$$\nabla \times \tilde{\mathbf{H}} e^{j\omega t} = j\omega\epsilon \tilde{\mathbf{E}} e^{j\omega t}, \tag{4}$$

where $e^{j\omega t}$ cancels out on both sides.
Equation (4) implies Equation (3). Also, any time dependence cancels out in the problem. Hence,

$$\nabla \times \tilde{\mathbf{H}} = j\omega\epsilon \tilde{\mathbf{E}}. \tag{5}$$

Similarly,

$$\nabla \times \tilde{\mathbf{E}} = -j\omega\mu \tilde{\mathbf{H}}, \tag{6}$$

$$\nabla \cdot \mu \tilde{\mathbf{H}} = 0, \tag{7}$$

$$\nabla \cdot \epsilon \tilde{\mathbf{E}} = 0. \tag{8}$$

Taking the curl of (6) and substituting (5) into it, we have

$$\nabla \times \nabla \times \tilde{\mathbf{E}} = -j\omega\mu \nabla \times \tilde{\mathbf{H}} = \omega^2 \mu\epsilon \tilde{\mathbf{E}}. \tag{9}$$

Again, making use of the identity $\nabla \times \nabla \times \tilde{\mathbf{E}} = \nabla(\nabla \cdot \tilde{\mathbf{E}}) - \nabla^2 \tilde{\mathbf{E}}$, and $\nabla \cdot \tilde{\mathbf{E}} = 0$, we have

$$\nabla^2 \tilde{\mathbf{E}} = -\omega^2 \mu\epsilon \tilde{\mathbf{E}}. \tag{10}$$

Similarly,

$$\nabla^2 \tilde{\mathbf{H}} = -\omega^2 \mu\epsilon \tilde{\mathbf{H}}. \tag{11}$$

These are the Helmholtz's wave equations.

Lossy Medium (Conductive Medium)

Phasor technique is particularly appropriate for solving Maxwell's equations in a lossy medium. In a lossy medium, Equation (3.1) becomes

$$\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}, \tag{12}$$

where $\mathbf{J}$ is the induced currents in the medium, and hence,

$$\mathbf{J} = \sigma \mathbf{E}. \tag{13}$$

Applying phasor technique to (12), we have

$$\begin{aligned} \nabla \times \tilde{\mathbf{H}} &= j\omega\epsilon\tilde{\mathbf{E}} + \sigma\tilde{\mathbf{E}} \\ &= j\omega \left(\epsilon - j\frac{\sigma}{\omega}\right) \tilde{\mathbf{E}}. \end{aligned} \tag{14}$$

We can define the quantity

$$\tilde{\epsilon} = \epsilon - j\frac{\sigma}{\omega} \tag{15}$$

to be the complex permittivity of the medium, and (14) becomes

$$\nabla \times \tilde{\mathbf{H}} = j\omega\tilde{\epsilon}\tilde{\mathbf{E}}. \tag{16}$$

Notice that the only difference between (16) and (5) is the complex permittivity versus the real permittivity. If one goes about deriving the Helmholtz wave equations for a lossy medium, the results are

$$\nabla^2 \tilde{\mathbf{E}} = -\omega^2 \mu\tilde{\epsilon}\tilde{\mathbf{E}}, \tag{17}$$

$$\nabla^2 \tilde{\mathbf{H}} = -\omega^2 \mu\tilde{\epsilon}\tilde{\mathbf{H}}. \tag{18}$$

Hence, a lossy medium is easily treated using phasor technique by replacing a real permittivity with a complex permittivity.

If we restrict ourselves to one dimension, Equation (17), for instance, becomes of the form

$$\frac{d^2}{dz^2}\tilde{E}_x(z) - \gamma^2 \tilde{E}_x(z) = 0, \tag{19}$$

where

$$\gamma = j\omega\sqrt{\mu\tilde{\epsilon}} = j\omega\sqrt{\mu\left(\epsilon - j\frac{\sigma}{\omega}\right)} = \alpha + j\beta. \tag{20}$$

The general solution to (19) is of the form

$$\tilde{E}_x(z) = C_1 e^{-\gamma z} + C_2 e^{+\gamma z}. \tag{21}$$

In real space time,

$$\begin{aligned} E_x(z,t) &= \Re e[\tilde{E}_x(z) e^{j\omega t}] \\ &= \Re e[C_1 e^{-\gamma z} e^{j\omega t}] + \Re e[C_2 e^{\gamma z} e^{j\omega t}] \end{aligned} \tag{23}$$

If $C_1 = |C_1| e^{j\phi_1}$, $\quad C_2 = |C_2| e^{j\phi_2}$, $\quad \gamma = \alpha + j\beta$, $\quad$ then

$$E_x(z,t) = |C_1| \cos(\omega t - \beta z + \phi_1) e^{-\alpha z} + \mid C_2 \mid \cos(\omega t + \beta z + \phi_2) e^{\alpha z}. \qquad (24)$$

Note that one of the solutions in (24) is decaying with z while another solution is growing with z. The function $\cos(\omega t \pm \beta z + \phi)$ can be written as $\cos[\pm\beta(z \pm \frac{\omega}{\beta} t) + \phi]$. Hence, it moves with a velocity

$$v = \frac{\omega}{\beta}. \qquad (25)$$

Depending on its sign, it moves either in the positive or negative z direction. In the above, γ is the ***propagation constant***, α is the ***attenuation constant*** while β is the ***phase constant***.

Intrinsic Impedance

The intrinsic impedance can be easily derived also in the phasor world. The phasor representation of Equation (3.23) is

$$\frac{d}{dz}\tilde{E}_x = -j\omega\mu\tilde{H}_y. \qquad (26)$$

A corresponding one for $\tilde{H}_y$ is

$$\frac{d}{dz}\tilde{H}_y = -j\omega\epsilon\tilde{E}_x. \qquad (27)$$

If we now let $\tilde{E}_x = E_0 e^{-\gamma z}$, $\tilde{H}_y = H_0 e^{-\gamma z}$, and using them in (26) yields

$$-\gamma E_0 e^{-\gamma z} = -j\omega\mu H_0 e^{-\gamma z}. \qquad (28)$$

The above implies that

$$\eta = \frac{E_0}{H_0} = \frac{j\omega\mu}{\gamma} = \sqrt{\frac{\mu}{\epsilon}}. \qquad (29)$$

For a lossy medium, we replace ϵ by the complex permittivity and the intrinsic impedance becomes

$$\eta = \sqrt{\frac{\mu}{\tilde{\epsilon}}} = \sqrt{\frac{\mu}{\epsilon - j\frac{\sigma}{\omega}}} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\mu}}. \qquad (30)$$

The above is obviously a complex number.