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ECE 350 Lecture Notes

3. Wave Equation from Maxwell's Equations

Lossless Medium

In a source free region, Maxwell's equations are

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t,} \tag{1}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{2}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{3}$$

$$\nabla \cdot \mathbf{D} = 0, \tag{4}$$

where $\mathbf{B} = \mu \mathbf{H}$ and $\mathbf{D} = \epsilon \mathbf{E}$. Taking the curl of (2), we have

$$\nabla \times (\nabla \times \mathbf{E}) = -\mu \frac{\partial}{\partial t} \nabla \times \mathbf{H}. \tag{5}$$

Substituting (1) into (5), we obtain

$$\nabla \times \nabla \times \mathbf{E} = -\mu\epsilon \frac{\partial^2}{\partial t^2} \mathbf{E}. \tag{6}$$

Making use of the vector identity that

$$\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}, \tag{7}$$

we have

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\mu\epsilon \frac{\partial^2}{\partial t^2} \mathbf{E}. \tag{8}$$

Since the region is source free, and $\nabla \cdot \mathbf{E} = 0$, we have

$$\nabla^2 \mathbf{E} = \mu\epsilon \frac{\partial^2}{\partial t^2} \mathbf{E}, \tag{9}$$

which is the vector wave equation in freespace where $\nabla \cdot \mathbf{E} = 0$.
Similarly, we can show that

$$\nabla^2 \mathbf{H} = \mu\epsilon \frac{\partial^2}{\partial t^2} \mathbf{H} \tag{10}$$

if $\nabla \cdot \mathbf{H} = 0$, which is, of course, true in free space.

Plane Wave Solutions to the Vector Wave Equations

The condition for arriving at Equation (9) is that $\nabla \cdot \mathbf{E} = 0$. We can have solutions of the form

$$\mathbf{E} = \hat{x} E_x(z,t), \tag{11}$$

$$\mathbf{E} = \hat{y} E_y(z,t), \tag{12}$$

but not

$$\mathbf{E} = \hat{z} E_z(z,t), \tag{13}$$

because (13) violates $\nabla \cdot \mathbf{E} = 0$ unless E_z is independent of z. If $\mathbf{E}$ is of the form (11), then

$$\nabla^2 \mathbf{E} = \hat{x} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) E_x(z,t) = \hat{x} \frac{\partial^2}{\partial z^2} E_x, \tag{14}$$

with both $\frac{\partial^2}{\partial x^2}$ and $\frac{\partial^2}{\partial y^2}$ equal to zero. Then (9) becomes

$$\frac{\partial^2}{\partial z^2} E_x(z,t) - \mu\epsilon \frac{\partial^2}{\partial t^2} E_x(z,t) = 0. \tag{15}$$

Similarly, if $\mathbf{H} = \hat{y} H_y(z,t)$, (10) becomes

$$\frac{\partial^2}{\partial z^2} H_y(z,t) - \mu\epsilon \frac{\partial^2}{\partial t^2} H_y(z,t) = 0. \tag{16}$$

Equations (15) and (16) are scalar, one dimensional wave equations of the form

$$\frac{\partial^2}{\partial z^2} y(z,t) - \frac{1}{v^2} \frac{\partial^2}{\partial t^2} y(z,t) = 0, \tag{17}$$

where $v = 1/\sqrt{\mu\epsilon}$. The solution to (17) is of the form $y = f(z + at)$. We can show that

$$\frac{\partial}{\partial z} f = f'(z + at), \qquad \frac{\partial f}{\partial t} = a f'(z + at), \tag{18}$$

$$\frac{\partial^2}{\partial z^2} f = f''(z + at), \qquad \frac{\partial^2 f}{\partial t^2} = a^2 f''(z + at). \tag{19}$$

Substituting (19) into (17), we have

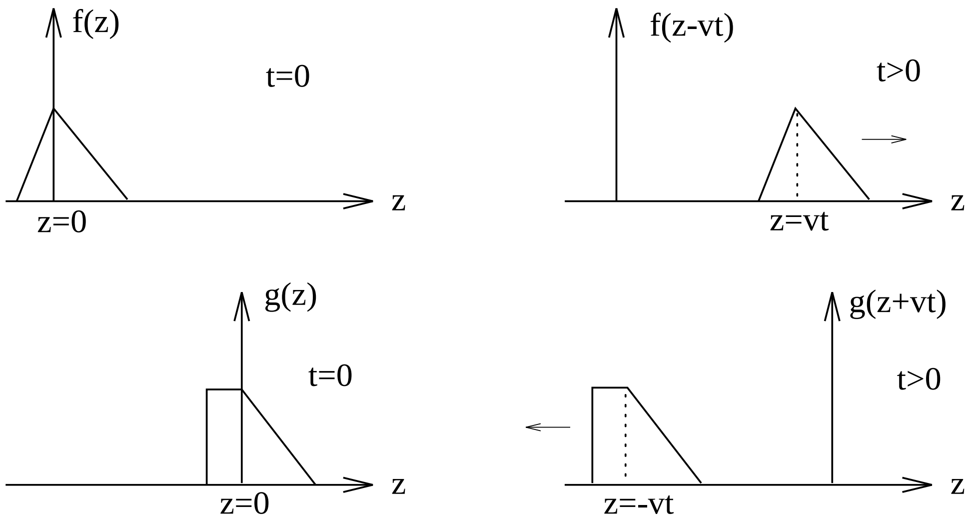
$$f''(z + at) - \frac{a^2}{v^2} f''(z + at) = 0, \tag{20}$$

which is possible only if $a = \pm v$. Hence, the general solution to the wave equation is

$$y = f(z - vt) + g(z + vt), \tag{21}$$

where f and g are arbitrary functions.

The solution $f(z - vt)$ moves in the positive z-direction for increasing t.



The solution $g(z + vt)$ moves in the negative z-direction for increasing t.

The shapes of the functions f and g are undistorted as they move along. We can observe wavelike behavior in a pond when we drop a pebble into it. Solutions to (9) and (10) that correspond to a plane wave is of the form

$$\mathbf{E} = \hat{x} f_1(z - vt), \qquad \mathbf{H} = \hat{y} f_2(z - vt). \tag{22}$$

The wave is propagating in the z-direction, but the electric and magnetic fields are transverse to the direction of propagation. Such a wave is known as the **T**ransverse **E**lectro **M**agnetic wave or TEM wave.

If one substitutes (22) into Equation (2), one has

$$\nabla \times \mathbf{E} = \hat{y} \frac{\partial}{\partial z} E_x = -\mu \frac{\partial}{\partial t} \mathbf{H}, \tag{23}$$

or

$$\frac{\partial}{\partial z} f_1(z - vt) = -\mu \frac{\partial}{\partial t} f_2(z - vt), \tag{24}$$

or

$$f_1'(z - vt) = \mu v f_2'(z - vt), \tag{25}$$

or

$$f_2(z - vt) = \sqrt{\frac{\epsilon}{\mu}} f_1(z - vt). \tag{26}$$

Hence, for a plane TEM wave,

$$\frac{E_x}{H_y} = \sqrt{\frac{\mu}{\epsilon}} = 377\,\Omega, \quad \text{for free space.} \tag{27}$$

The quantity

$$Z = \sqrt{\frac{\mu}{\epsilon}} \tag{28}$$

is also known as the intrinsic impedance of free-space.