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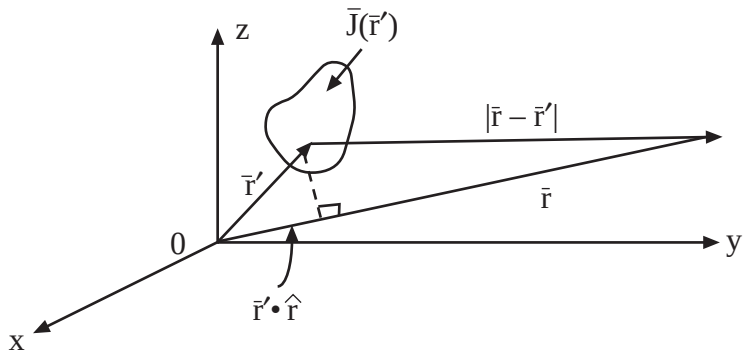
ECE 350 Lecture Notes

27. Radiation Field Approximations

The vector potential due to a source $\mathbf{J}(\mathbf{r})$, can be calculated from the equation

$$\mathbf{A}(\mathbf{r}) = \iiint_V d\mathbf{r}' \frac{\mu \mathbf{J}(\mathbf{r}')}{4\pi\,|\mathbf{r}-\mathbf{r}'|} e^{-j\beta|\mathbf{r}-\mathbf{r}'|}, \tag{1}$$

where V is the volume occupied by $\mathbf{J}(\mathbf{r})$.



When $|\mathbf{r}| \gg |\mathbf{r}'|$, then $|\mathbf{r}-\mathbf{r}'| = r - \mathbf{r}'\cdot\hat{r}$. Equation (1) becomes

$$\begin{aligned}
\mathbf{A}(\mathbf{r}) &\cong \iiint_V d\mathbf{r}' \frac{\mu \mathbf{J}(\mathbf{r}')}{r - \mathbf{r}'\cdot\hat{r}} e^{-j\beta r} e^{j\beta \mathbf{r}'\cdot\hat{r}} \\
&= \frac{\mu e^{-j\beta r}}{4\pi r} \iiint_V d\mathbf{r}' \mathbf{J}(\mathbf{r}') e^{j\beta\mathbf{r}'\cdot\hat{r}} \\
&= e^{-j\beta r}\frac{\mathbf{f}(\theta,\phi)}{r} = \hat{\theta}A_\theta + \hat{\phi}A_\phi + \hat{r}A_r.
\end{aligned} \tag{2}$$

In the above we have assumed that $|\mathbf{r}'\cdot\hat{r}| \ll r$ but $\beta\mathbf{r}'\cdot\hat{r}$ is not small, since β can be large. When βr is large, $\frac{f(\theta,\phi)}{r}$ is a slowly varying function compared to $e^{-j\beta r}$. Hence, we can regard $\frac{\mathbf{f}(\theta,\phi)}{r}$ almost to be a constant compared to $e^{-j\beta r}$. The magnetic field can be derived to be

$$\mathbf{H} = \frac{1}{\mu}\nabla\times\mathbf{A} \approx -\frac{1}{\mu}\left[\hat{\theta}\frac{\partial}{\partial r}A_\phi - \hat{\phi}\frac{\partial}{\partial r}A_\theta\right]. \tag{3}$$

However, $\frac{\partial}{\partial r} \sim -j\beta$ when βr is large. Hence,

$$\mathbf{H} = \frac{j\beta}{\mu}(\hat{\theta}A_\phi - \hat{\phi}A_\theta), \qquad \text{when } \beta r \to \infty. \tag{4}$$

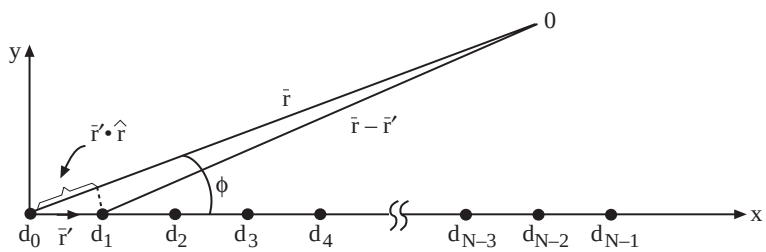
Similarly,

$$\mathbf{E} = \frac{1}{j\omega\epsilon}\nabla \times \mathbf{H} \cong -j\omega[\hat{\theta}A_\theta + \hat{\phi}A_\phi]. \tag{5}$$

Linear Array of Dipole Antennas

If $\mathbf{J}(\mathbf{r}')$ is of the form

$$\begin{aligned}\mathbf{J}(\mathbf{r}') = \hat{z}Il[A_0\delta(x') + A_1\delta(x'-d_1) + A_2\delta(x'-d_2) \\ + \cdots + A_{N-1}\delta(x'-d_{N-1})]\delta(y')\delta(z'),\end{aligned} \tag{6}$$



the vector potential on the xy-plane can be derived to be

$$\begin{aligned}\mathbf{A}(\mathbf{r}) &= \hat{z}\frac{\mu Il}{4\pi r}e^{j\beta r}\iiint d\mathbf{r}'[A_0\delta(x') + A_1\delta(x'-d_1) + \cdots]\delta(y')\delta(z')e^{+j\beta\mathbf{r}'\cdot\hat{r}} \\ &= \hat{z}\frac{\mu Il}{4\pi r}e^{-j\beta r}\left[A_0 + A_1e^{+j\beta d_1\cos\phi} + A_2e^{j\beta d_2\cos\phi} + \cdots + A_{N-1}e^{j\beta d_{N-1}\cos\phi}\right].\end{aligned} \tag{7}$$

If $d_n = nd$, and $A_n = e^{jn\psi}$, then (7) becomes

$$\mathbf{A}(\mathbf{r}) = \hat{z}\frac{\mu Il}{4\pi r}e^{-j\beta r}\left[1 + e^{j(\beta d\cos\phi+\psi)} + e^{2j(\beta d\cos\phi+\psi)} + \cdots + e^{j(N-1)(\beta d\cos\phi+\psi)}\right], \tag{8}$$

which is of the form

$$\sum_{n=0}^{N-1} x^n = \frac{1-x^N}{1-x}. \tag{9}$$

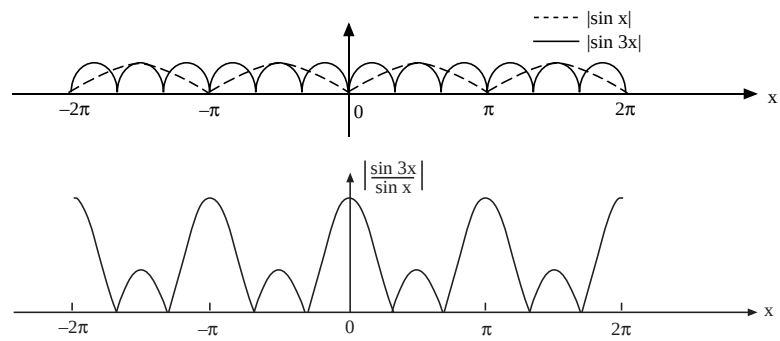
Therefore,

$$\mathbf{A}(\mathbf{r}) = \hat{z}\frac{\mu Il}{4\pi r}e^{-j\beta r}\frac{1-e^{jN(\beta d\cos\phi+\psi)}}{1-e^{j(\beta d\cos\phi+\psi)}}. \tag{10}$$

The electric field on the xy-plane is $E_\theta = -j\omega A_\theta = +j\omega A_z$. Hence, $|E_\theta|$ is of the form

$$\begin{aligned}|E_\theta| &= |E_0|\left|\frac{1-e^{jN(\beta d\cos\phi+\psi)}}{1-e^{j(\beta d\cos\phi+\psi)}}\right| \\ &= |E_0|\left|\frac{\sin\frac{N}{2}(\beta d\cos\phi+\psi)}{\sin\frac{1}{2}(\beta d\cos\phi+\psi)}\right|.\end{aligned} \tag{11}$$

Equation (11) is of the form $\frac{|\sin Nx|}{|\sin x|}$. Plots of $|\sin 3x|$ and $|\sin x|$ are shown as an example.



In equation (11), $\lambda = \frac{1}{2}(\beta d \cos\phi + \psi)$. We notice that the **maximum** in (11) would occur if $\lambda = n\pi$, or if

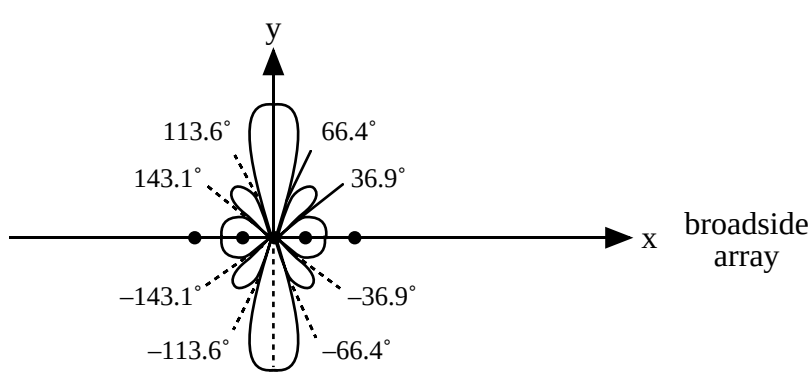
$$\beta d \cos\phi + \psi = 2n\pi, \qquad n = 0, \pm 1, \pm 2, \pm 3, \cdots . \tag{12}$$

The **zeros** or **nulls** will occur at $Nx = n\pi$, or

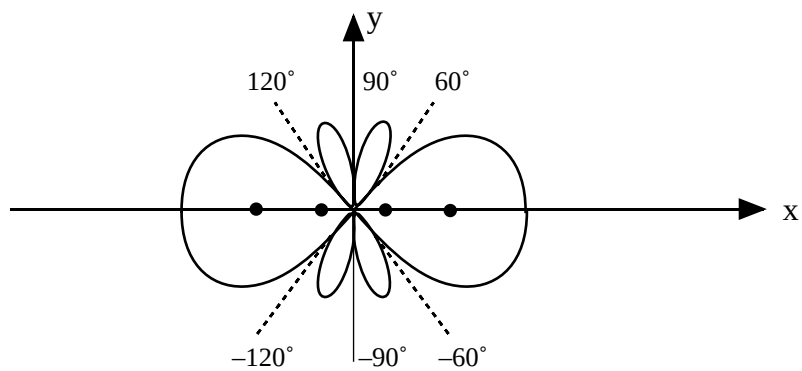
$$\beta d \cos\phi + \psi = \frac{2n\pi}{N}, \qquad n = \pm 1, \pm 2, \pm 3, \cdots , \quad n \neq mN. \tag{13}$$

For example,

Case I. $\psi = 0, \beta d = \pi$, principal maximum is at $\phi = \pm\frac{\pi}{2}$ if $N = 5$, nulls are at $\phi = \pm\cos^{-1}\left(\frac{2n}{5}\right)$, or $\phi = \pm 66.4^\circ, \pm 36.9^\circ, \pm 113.6^\circ, \pm 143.1^\circ$.



Case II. $\psi = \pi, \beta d = \pi$, principal maximum is at $\phi = 0, \pi$, if $N = 4$, nulls are at $\phi = \pm\cos^{-1}\left(\frac{n}{2} - 1\right)$, or $\phi = \pm 120^\circ, \pm 90^\circ, \pm 60^\circ$.



The interference effects between the different antenna elements of a linear array focus the power in a given direction. We can use linear array to increase the directivity of antennas.

Note that equation (7) can also be derived by other means. We know that the vector potential due to one dipole is

$$\mathbf{A} = \hat{z}\frac{\mu I l}{4\pi}\frac{e^{-j\beta|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}, \tag{14}$$

when the dipole is located at $\mathbf{r}'$ and pointing in the $\hat{z}$-direction. Hence for an array of dipoles of different phases and amplitudes, located at $x = \hat{x}d_0, \hat{x}d_1, \hat{x}d_2, \cdots, \hat{x}d_{N-1}$, the vector potential by linear superposition is

$$\mathbf{A}(\mathbf{r}) = \hat{z}\frac{\mu I l}{4\pi}\left[\frac{e^{-j\beta|\mathbf{r}-\hat{x}d_0|}}{|\mathbf{r}-\hat{x}d_0|}A_0 + \frac{e^{-j\beta|\mathbf{r}-\hat{x}d_1|}}{|\mathbf{r}-\hat{x}d_1|}A_1 + \cdots + \frac{e^{-j\beta|\mathbf{r}-\hat{x}d_{N-1}|}}{|\mathbf{r}-\hat{x}d_{N-1}|}A_{N-1}\right]. \tag{15}$$

If we approximate $|\mathbf{r}-\hat{x}d_n|$ by $r - \hat{r}\cdot\hat{x}d_N = r - d_N\cos\phi$, in the phase, and by r in the denominator, then (15) becomes

$$\mathbf{A}(\mathbf{r}) = \hat{z}\frac{\mu I l}{4\pi r}e^{-j\beta r}\left[A_0 + A_1 e^{+j\beta d_1\cos\phi} + A_2 e^{j\beta d_2\cos\phi} + \cdots + A_{N-1}e^{j\beta d_{N-1}\cos\phi}\right], \tag{16}$$

which is the same as equation (7). The interference between the terms in (16) can be used to generate different radiation patterns for different communication applications.