The Recursive Aggregate Interaction Matrix Algorithm for Multiple Scatterers

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Abstract

The recursive aggregate interaction matrix algorithm (RAIMA) for calculating wave scattering solution is developed. This algorithm combines the strength of both the recursive aggregate T matrix algorithm (RATMA) and the recursive interaction matrix algorithm (RIMA) that have been previously developed. The resultant algorithm is robust for scattering problems involving highly singular Green's function by avoiding the violation of the addition theorem. It also has reduced computational complexity for inverting the volume integral equation of scattering operator. The computational complexity of RAIMA is $O(N^{7/3})$ in three dimensions and $O(N^2)$ in two dimensions.

1. Introduction

Fast and efficient methods to solve electromagnetic scattering problems have been topics of active research interest recently [1-10]. One of these methods, the recursive aggregate T matrix algorithm (RATMA) has been developed to solve directly (instead of iteratively) for the scattering solution of N subscatterers with reduced computational complexity [11-15]. For a general scattering problem, a scatterer is first decomposed into N subscatterers which are non-overlapping before RATMA is applied.

However, the close proximity of the subscatterers gives rise to the violation of the addition theorem when RATMA is applied. Several remedies have been proposed to alleviate the error arising from the violation of the addition

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theorem [12,13]. We will report here yet another method to remove the problem associated with violating the addition theorem. The method described here will even allow the subscatterers to be overlapping, so that it could be applied to solve matrices from the method of moment (MOM) [16-17].

This improved algorithm, the recursive aggregate interaction matrix algorithm (RAIMA) is a modification of the recursive aggregate T matrix algorithm (RATMA) to take advantage of the recursive interaction matrix algorithm (RIMA) [19]. RIMA can be used to invert a method of moments (MOM) matrix, but with $O(N^3)$ complexity. Incorporating some ideas of RATMA into RIMA, one can arrive at an algorithm with reduced computational complexity for tightly packed or overlapping subscatterers.

First we shall show describe RIMA. Then, we will derive RAIMA from RIMA by aggregation. This will eventually yield an algorithm with reduced complexity.

2. The Recursive Interaction Matrix Algorithm (RIMA)

Given a scatterer, it can be decomposed into N subscatterers, and treated as an N-scatterer problem. Then, the total field can be written as [15]

$$\mathbf{E}(\mathbf{r}) = \boldsymbol{\psi}^t(k_0, \mathbf{r}_s) \cdot \mathbf{a}_s + \sum_{i=1}^{N} \boldsymbol{\psi}^t(k_0, \mathbf{r}_i) \cdot \sum_{j=1}^{N} \overline{\mathbf{I}}_{ij(N)} \cdot \overline{\boldsymbol{\alpha}}_{js} \cdot \mathbf{a}_s, \tag{1}$$

where $\mathbf{r}_i = \mathbf{r} - \mathbf{r}'_i$, and $\mathbf{r}'_i$ is the location of the center of the i-th subscatterer. The row vector $\boldsymbol{\psi}^t(k_0, \mathbf{r}_i)$ contains the cylindrical or spherical harmonics. The first term in (1) is from the source, while the second term is the scattered field from the N subscatterers. The vector $\mathbf{a}_s$ contains the amplitudes of the harmonics for the source term, while $\overline{\mathbf{I}}_{ij(N)}$ is the interaction matrix for N subscatterers, and $\overline{\boldsymbol{\alpha}}_{js}$ is a translation matrix obtainable from the addition theorem [12,20]. A recursive algorithm can be derived so that the interaction matrices for $(n+1)$ subscatterers, $\overline{\mathbf{I}}_{ij(n+1)}$, can be derived from those for n subscatterers, $\overline{\mathbf{I}}_{ij(n)}$. The interaction matrices are of dimension $M \times M$ where

M is the number of harmonics needed to approximate the scattered field from each subscatterer well. The recursive equations are [15,19]

$$\overline{\mathbf{I}}_{n+1,n+1(n+1)} = \left[\overline{\mathbf{I}} - \overline{\mathbf{T}}_{n+1(1)} \cdot \sum_{i=1}^{n} \sum_{j=1}^{n} \overline{\boldsymbol{\alpha}}_{n+1,i} \cdot \overline{\mathbf{I}}_{ij(n)} \cdot \overline{\boldsymbol{\alpha}}_{j,n+1} \right]^{-1} \cdot \overline{\mathbf{T}}_{n+(1)}. \quad \text{(2a)}$$

$$\overline{\mathbf{I}}_{n+1,j(n+1)} = \overline{\mathbf{I}}_{n+1,n+1(n+1)} \cdot \left[\sum_{i=1}^{n} \overline{\boldsymbol{\alpha}}_{n+1,i} \cdot \overline{\mathbf{I}}_{ij(n)} \right], \qquad j = 1, \cdots, n. \quad \text{(2b)}$$

$$\overline{\mathbf{I}}_{i,n+1(n+1)} = \left[\sum_{j=1}^{n} \overline{\mathbf{I}}_{ij(n)} \cdot \overline{\boldsymbol{\alpha}}_{j,n+1} \right] \cdot \overline{\mathbf{I}}_{n+1,n+1(n+1)} \qquad i = 1, \cdots, n, \quad \text{(2c)}$$

$$\overline{\mathbf{I}}_{ij(n+1)} = \overline{\mathbf{I}}_{ij(n)} + \left[\sum_{j'=1}^{n} \overline{\mathbf{I}}_{ij'(n)} \cdot \overline{\boldsymbol{\alpha}}_{j',n+1} \right] \cdot \overline{\mathbf{I}}_{n+1,j(n+1)}, \qquad i, j = 1, \cdots, n. \quad \text{(2d)}$$

The above equations require $O(n^2)$ operations per step to implement and after N recursive steps, $O(N^3)$ operations are required.

3. The Recursive Aggregate Interaction Matrix Algorithm —(RAIMA)

Since RIMA requires $O(N^3)$ operations, it is no better than Gaussian elimination. To expedite the computation, the interaction matrix for n subscatterers has to be aggregated into that for one larger subscatterer. This results in a reduced representation for the scattered field for n subscatterers. By doing so, we arrive at the recursive aggregate T matrix algorithm (RATMA). However, in RATMA, there is a potential violation of the addition theorem. Several methods have been proposed to mitigate the error that arises from the violation of the addition theorem [12,13]. Here, we propose yet another approach to minimize this error.

In this new approach, we divide the subscatterers into n' subscatterers in the buffer zone, and $(n - n')$ subscatterers in the inner zone. We would only aggregate the $(n - n')$ subscatterers in the inner zone to expedite the computational efficiency. We shall call this the recursive aggregate interaction matrix algorithm (RAIMA). It has the same computational complexity as RATMA.

To this end, we define an aggregate interaction matrix for $(n-n')$ subscatterers as

$$\overline{\mathbf{I}}^{a}_{n-n',n-n'(n)} = \sum_{i=1}^{n-n'} \sum_{j=1}^{n-n'} \overline{\boldsymbol{\beta}}_{0i} \cdot \overline{\mathbf{I}}_{ij(n)} \cdot \overline{\boldsymbol{\beta}}_{j0}, \tag{3a}$$

$$\overline{\mathbf{I}}^{a}_{i,n-n'(n)} = \sum_{j=1}^{n-n'} \overline{\mathbf{I}}_{ij(n)} \cdot \overline{\boldsymbol{\beta}}_{j0}, \qquad i = n-n'+1, \cdots, n, \tag{3b}$$

$$\overline{\mathbf{I}}^{a}_{n-n',j(n)} = \sum_{j=1}^{n-n'} \overline{\boldsymbol{\beta}}_{0i} \cdot \overline{\mathbf{I}}_{ij(n)}, \qquad j = n-n'+1, \cdots, n, \tag{3c}$$

Using the fact that [20]

$$\overline{\boldsymbol{\alpha}}_{n+1,i} = \overline{\boldsymbol{\alpha}}_{n+1,0} \cdot \overline{\boldsymbol{\beta}}_{0i}, \quad r_{n+1} > r_i, \tag{4a}$$

$$\overline{\boldsymbol{\alpha}}_{j,n+1} = \overline{\boldsymbol{\beta}}_{j,0} \cdot \overline{\boldsymbol{\alpha}}_{0,n+1}, \quad r_{n+1} > r_j, \tag{4b}$$

Equations (2a) to (2d) can be rewritten for $n'+1$ subscatterers. By using (4a) and (4b), and then (3) in (2a), we have

$$\overline{\mathbf{I}}_{n+1,n+1} = \left[\overline{\mathbf{I}} - \overline{\mathbf{T}}_{n+1(1)} \cdot \sum_{i=n-n'}^{n} \overline{\boldsymbol{\alpha}}_{n+1,i} \cdot \overline{\mathbf{I}}_{ij(n)} \cdot \overline{\boldsymbol{\alpha}}_{j,n+1} \right]^{-1} \cdot \overline{\mathbf{T}}_{n+1(1)}, \tag{5}$$

where $\overline{\boldsymbol{\alpha}}_{n+1,n-n'} = \overline{\boldsymbol{\alpha}}_{n+1,0}$, $\overline{\boldsymbol{\alpha}}_{n-n',n+1} = \overline{\boldsymbol{\alpha}}_{0,n+1}$, and when either i or j equals $n-n'$, $\overline{\mathbf{I}}_{ij(n)} = \overline{\mathbf{I}}^{(a)}_{ij(n)}$ defined in Equation (3)

Right multiplying (2b) by $\overline{\boldsymbol{\beta}}_{j,0}$ and sum over j from 1 to $n-n'$, and using (4a) in (2b) for $i \leq n-n'$, we have

$$\overline{\mathbf{I}}_{n+1,j(n+1)} = \overline{\mathbf{I}}_{n+1,n+1(n+1)} \cdot \left[\sum_{i=n-n'}^{n} \overline{\boldsymbol{\alpha}}_{n+1,i} \cdot \overline{\mathbf{I}}_{ij(n)} \right], \qquad j = n-n', \cdots, n. \tag{6}$$

Similar treatment to (2c) yields

$$\overline{\mathbf{I}}_{i,n+1(n+1)} = \left[\sum_{j=n-n'}^{n} \overline{\mathbf{I}}_{ij(n)} \cdot \overline{\boldsymbol{\alpha}}_{j,n+1} \right] \cdot \overline{\mathbf{I}}_{n+1,n+1(n+1)}, \qquad i = n-n', \cdots, n. \quad (7)$$

Left and right multiplying (2d) by $\overline{\boldsymbol{\beta}}_{0i}$ and $\overline{\boldsymbol{\beta}}_{j0}$ respectively and sum over i and j from 1 to $n-n'$, we arrive at

$$\overline{\mathbf{I}}_{i,j(n+1)} = \overline{\mathbf{I}}_{ij(n)} + \left[\sum_{j'=n-n'}^{n} \overline{\mathbf{I}}_{ij(n)} \cdot \overline{\boldsymbol{\alpha}}_{j,n+1} \right] \cdot \overline{\mathbf{I}}_{n+1,j(n+1)}, \qquad i,j = n-n', \cdots, n. \quad (8)$$

The same remark about $\overline{\mathbf{I}}_{ij(n)}$ after Equation (5) applies to (6), (7) and (8). Equations (5) to (6) are the rewrites of Equation (2a) to (2d) where the $(n-n')$ inner subscatterers are replaced by an aggregate subscatterer.

As new subscatterers are added in the recursive algorithm, the size of the aggregate subscatterer needs to be enlarged so that the size of the buffer zone remains minimal. When some subscatterers are shifted from the buffer zone to the inner zone, Equation (3) can be used to derive an update equation for $\overline{\mathbf{I}}^{a}_{n-n',n-n'(n)}$. The aggregate interaction matrix for inner zone having $(n-n'+l)$ subscatterers can be derived from that for $(n-n')$ subscatterers via.

$$\overline{\mathbf{I}}^{a}_{n-n'+l,n-n'+l(n+1)} = \sum_{i=n-n'}^{n-n'+l} \overline{\boldsymbol{\beta}}_{0i} \cdot \sum_{j=n-n'}^{n-n'+l} \overline{\mathbf{I}}_{ij(n+1)} \cdot \overline{\boldsymbol{\beta}}_{j0}, \quad (9a)$$

$$\overline{\mathbf{I}}^{a}_{i,n-n'+l(n+1)} = \sum_{j=n-n'}^{n-n'+l} \overline{\mathbf{I}}_{ij(n+1)} \cdot \overline{\boldsymbol{\beta}}_{j0}, \qquad i = n-n'+l+1, \cdots, n+1, \quad (9b)$$

$$\overline{\mathbf{I}}^{a}_{n-n'+l,j(n+1)} = \sum_{i=n-n'}^{n-n'+l} \overline{\boldsymbol{\beta}}_{0i} \cdot \overline{\mathbf{I}}_{ij(n+1)}, \qquad j = n-n'+l+1, \cdots, n+1, \quad (9c)$$

where $\overline{\boldsymbol{\beta}}_{0,n-n'} = \overline{\boldsymbol{\beta}}_{n-n',0} = 1$. The above constitute the RAIMA equations. When the size of the buffer zone is reduced to zero, it is the some as RATMA.

Once the interaction matrices are found, Equation (1) can be used to find the scattered field. When the interaction matrices are aggregated into a interaction matrix for the inner zone, only a reduced number of interaction matrices is needed to calculate the scattered field from Equation (1).

4. Computational Complexity

The recursive aggregate interaction matrix algorithm progressively aggregates a portion of the subscatterers into a larger aggregated subscatterer as shown in Figure 1. By so doing, one needs to contend with $n' + 1$ subscatterers rather than n subscatterers where $n' \ll n$. This is achieved at the expense of making the interaction matrix associated with the aggregated subscatterer larger. For instance, the interaction matrix defined by (3a) for the aggregated subscatterer is a $P \times P$ matrix (instead of $M \times M$ for the original subscatterers) where P is the number of harmonics that will make the factorization in (4a) and (4b) accurate. An analysis of the computational cost shows that if $P < n - n'$, a saving in computation time ensues.

For volumetric scatterers in three dimensions, the number of spherical harmonics needed to accurately expand the field around a scatterer of diameter d is proportional to $(kd)^2$ or the surface area of the circumscribing sphere that encloses the scatterer. In aggregating the subscatterers, what we have done is to describe the scattering processes associated with the aggregated subscatterers via the spherical harmonics rather than via the subscatterers directly. For instance, if there are $n - n'$ subscatterers, an interaction matrix (or a MOM matrix) description of the full scattering processes will require at least an $(n - n') \times (n - n')$ matrix, whereas a description via the spherical harmonics will entail a $P \times P$ matrix. Since $P \sim (kd)^2$ while $n - n' \sim (kd)^3$, $P \ll n - n'$ when the scattering problem is large.

The process of aggregation essentially replaces an n-unknown problem with a problem involving $n' + P$ unknown problem at each recursive step. The recursive interaction matrix solution to the original problem requires $O(n^2)$

operations per step, while for the aggregated problem, it requires $O((n' + P)^2)$ operations. Therefore, after N steps required to solve the problem, the original problem requires $O(N^3)$ operations while the aggregated problem will require $O(N(n'+P)^2)$ operations. If the buffer zone as shown in Figure 1 is of a fixed thickness, then $n' \sim (kd)^2$ as well (just like P). Then, $n' + P \sim (kd)^2$. Assuming that $N \sim (kd)^3$,* then $n'+P \sim N^{2/3}$, and the overall complexity of the algorithm is $O(N^{7/3})$. In two dimensions, the computational complexity can be shown to be $O(N^2)$.

A set of back-recursion formulas can be derived to calculate the internal field of the scatterer just as in [11]. If only the scattered field is needed, the memory requirement of this algorithm is $O(N)$ in two dimensions, and $O(N^{4/3})$ in three dimensions. If internal field is needed, so that a back-recursion algorithm needs to be invoked, the memory requirement is $O(N^{3/2})$ in two dimensions, and $O(N^{5/3})$ in three dimensions.

In general, for surface scatterers where the scattered field can be attributed to induced current on the surface of the scatterer, the aforementioned method does not reduce the computational complexity, but will reduce the computational time. However, this method has no restriction on the shape of the geometry.

5. Numerical Results and Conclusions

We have applied this algorithm to calculate the scattering solution of H_z wave scattering by metallic scatterers in two dimensions. For this polarization, the current flows in the transverse to z plane, and accurate computation of the scattering solution requires overlapping current basis functions [17]. A direct use of RATMA to this problem gives rise to poor result because of the absence of the buffer zone to prevent the violation of the addition theorem. With RAIMA, the problem is rectified. (The above algorithm can be easily

* This is a reasonable assumption, with our present knowledge of the problem. A better knowledge of how discretization density scales as the size of the problem may alter this assumption.

modified for overlapping subscatterers.) Figure 2 illustrates the computation of the RCS of a metallic scatterer (where the subscatterers are overlapping) consisting of a flat strip of width 2λ. The incident wave is H_z polarized from 45^o direction. There are five curves corresponding to $n' = 0, 1, 2, 5$ and the exact solution, RIMA. RIMA is equivalent to solving the matrix equation with Gaussian elimination.

This new algorithm can be used to improve the accuracy in calculating the multiple scattering solutions of discrete particles especially when the particles are large compared to wavelength and tightly packed together. For this cases, the violation or the ill-convergence of the addition theorem usually gives rise to numerical instabilities. This problem is rectified by RAIMA which includes a buffer zone that precludes this infraction. Figure 3 shows the scattered field of 200 randomly distributed spheres. The spheres are of the size $ka = 1.0$ and permittivity of $\epsilon_r = 3.2$ and have a fractional volume of 25%. The incident wave propagates in the direction with $\theta = 90^0$ and $\phi = 0^0$. The observation circle is in the xy plane which is 10λ away from the scattering center. (a) shows the field magnitude while (b) shows the field phase.

Figure 4 shows the CPU time (on a SUN Workstation) as a function of the number of spheres. Each sphere is of size $ka = 1.0$ and permittivity $\epsilon_r = 3.2$ and 6 modes are used in the expansion of the scattering field. The spheres are distributed randomly at a fractional volume of 25%. The dotted line is a curve for solving the problem using Gaussian elimination.

In conclusion, we have developed an improved algorithm that removes the problem associated with the violation of the addition theorem of the earlier algorithms. This violation of the addition theorem is a severe problem when the Green's function is highly singular. This is encountered in three-dimensional electromagnetic scattering problems, and two-dimensional scattering problems involving H_z polarization. Our results show that the new algorithm can be applied to densely packed scatterers in three dimensions, and even to overlapping subscatterers. Hence, it can be used to invert the method of moments matrix with reduced computational complexity.

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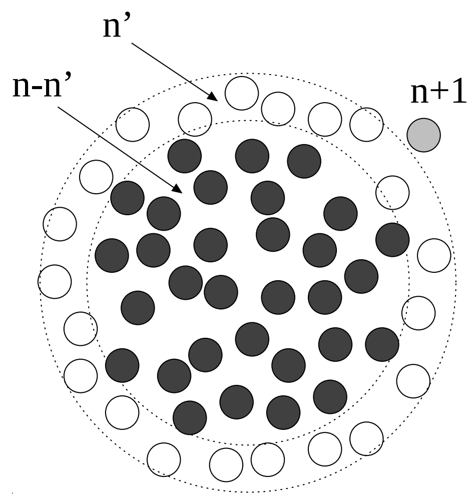

Figure 1 A cluster of n subscatterers is divided into a buffer zone of n' subscatterers to avoid the violation of the addition theorem. The $n - n'$ subscatterers are aggregated into one larger subscatterer more efficiently described by spherical harmonics.

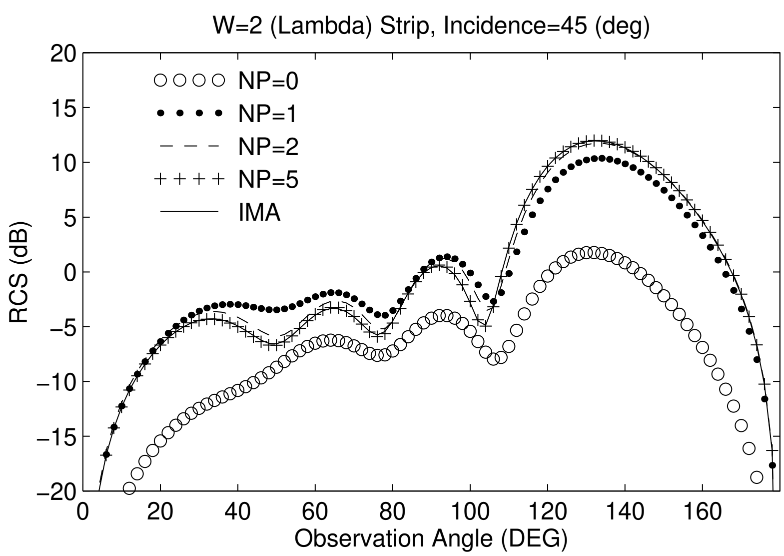

Figure 2 Numerical computation of the RCS of a metallic scattering of H_z polarized waves in two dimensions. This scattering problem can be solved by the method of moments using overlapping basis functions, and also by RAIMA more efficiently. The recursive interaction matrix approach (RIMA) is equivalent to Gaussian elimination. The strip is 2λ wide, and the incident angle is 45^o. Different n' (shown as NP in the Figure) values are used as the buffer zone.

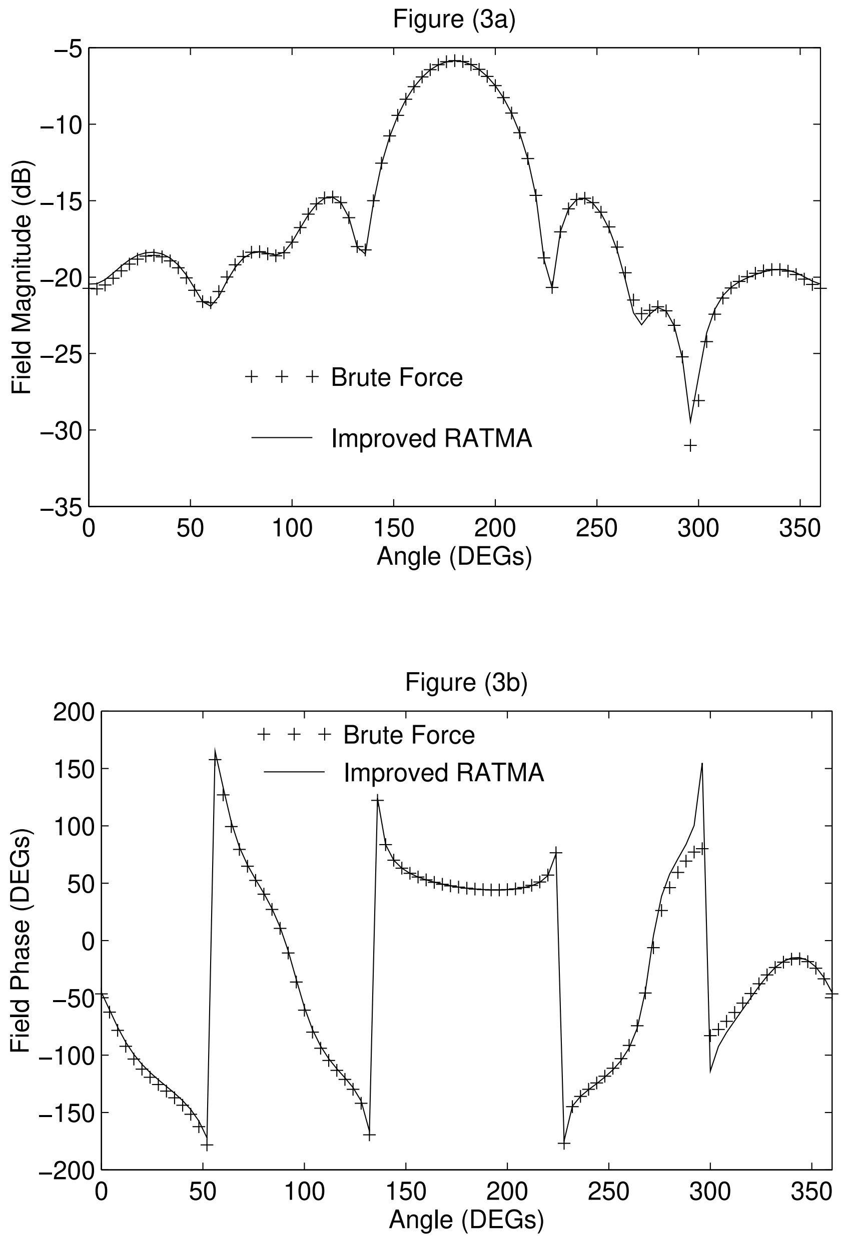

Figure 3 Numerical simulation of 200 randomly distributed spheres in three dimensions using RAIMA. The solid line is brute-force matrix inversion, while the dotted line is from RAIMA. RATMA fails to produce result of similar accuracy for this case where $ka = 1$, $\epsilon_r = 3.2$, and the fraction volume is 25%. Forty spheres are used as the buffer zone here. (a) Field magnitude; (b) Field phase.

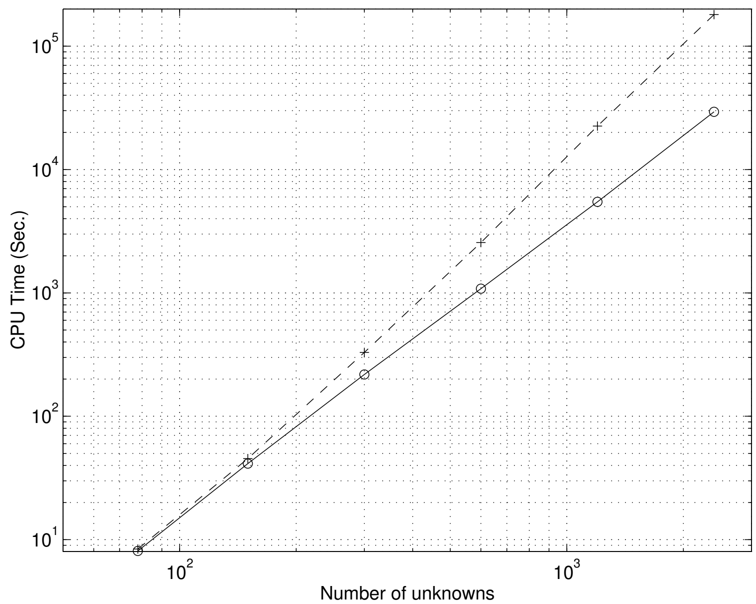

Figure 4 The growth of CPU time as a function of the number of unknowns in the three dimensional simulation of Figure 3. The solid line is for RAIMA while the dotted line is for Gaussian elimination.