Electromagnetic Theory on a Lattice

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Abstract

In this paper, a self-contained electromagnetic theory is derived on a regular lattice. The discretized form of integral and differential calculus, which is called 'discrete calculus' here, is used to describe this theory. It is shown that discrete forms of Gauss' theorem, Stokes' theorem, Green's theorem, and Huygens' principle can be derived. Moreover, many electromagnetic theorems can also be derived in this discretized world, for example, reciprocity theorem, uniqueness theorem, and Poynting's theorem. The preservation of these theorems and the conservation of charge imply that the use of this discretized form of Maxwell's equations for numerical simulation will not give rise to spurious solutions due to spurious charges.

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1. Introduction

Due to the advances in computer technology, there has been much interest in solving physical problems in the discretized world [1-5]. These have mainly been achieved by replacing differential equations with difference equations. The reason is that computers can only digest data that are inherently discrete. However, the laws of physics have been first written using partial differential equations and integrals, which suggest a continuum description of our world. The use of continuum calculus to describe physical laws may have been the result of its convenience and beauty. It is of interest to investigate if an equally beautiful description of the physical laws can be achieved with difference equations.

Electromagnetic theory has been described with differential and integral calculus, which assumes that space is a continuum and is infinitely divisible. However, experimental measurements have not detected electromagnetic radiation shorter in wavelength than that of the gamma ray. Hence, it is of interest to investigate if a self-consistent and self-contained discrete electromagnetic theory can be written on a lattice. Such a theory has been written in the past [6]. There are many attempts to formulate such theory with a more compact notation [7-11]. We will augment such work by showing that many theorems that exist for the continuum theory also exist for the discrete electromagnetic theory. These include the reciprocity theorem, Poynting's theorem, uniqueness theorem and Huygens' principle. The existence of these theorems in discretized forms also implies that a discretized form of electromagnetic theory can be used to predict and explain all classical electromagnetic phenomena observed so far as long as the discretization lengthscale is chosen small enough. These theorems are also useful in the finite difference simulation of Maxwell's equations. As will be shown, because of its charge conserving property, this discretized form of Maxwell's theory will not give rise to spurious solution due to spurious charges.

It is to be pointed out that electromagnetic theory can also be discretized by the finite element method [12]. It is easy to show that the reciprocity theorem is preserved under finite element discretization. However, it is not easy to show that Poynting's theorem, Huygens' principle, and Green's theorem are preserved under finite element discretization.

To this end, we will develop a calculus on a discretized world that can be used to describe the discretized electromagnetic theory. We shall call this "discrete calculus."

2. One-Dimensional Discrete Calculus

Continuum derivatives can be replaced by finite differences. We shall first derive some simple formulas and introduce some new notations for finite differences.

2.1. Difference

A differentiation can be replaced by a forward difference or a backward difference. If we have $g(x) = \partial_x f(x)$, its analog in the discrete space is

$$g_{m+\frac{1}{2}} = \tilde{\partial}_x f_m = \frac{1}{\Delta_x}[f_{m+1} - f_m], \tag{1}$$

or

$$g_{m-\frac{1}{2}} = \hat{\partial}_x f_m = \frac{1}{\Delta_x}[f_m - f_{m-1}], \tag{2}$$

where $f_m = f(m\Delta_x)$, Δ_x is the discretization size, and $\tilde{\partial}_x$ or $\hat{\partial}_x$ are the analogs of ∂_x in the continuous case. $g_{m+\frac{1}{2}}$ and $g_{m-\frac{1}{2}}$ are defined at half-grid points. Note that

$$\tilde{\partial}_x f_{m-1} = \hat{\partial}_x f_m, \qquad \mathcal{Z}\hat{\partial}_x f_m = \tilde{\partial}_x f_m, \tag{3}$$

where $\mathcal{Z}$ is a shifting operator such that $\mathcal{Z} f_m = f_{m+1}$. Also, note that $\hat{\partial}_x \tilde{\partial}_x = \tilde{\partial}_x \hat{\partial}_x$, i.e., they commute.

2.2. Discrete Chain Rule

An analog of chain rule can be derived for the finite difference formulas. It can be shown that

$$\begin{aligned}\tilde{\partial}_x(a_m b_m) &= \frac{1}{\Delta_x}(a_{m+1}b_{m+1} - a_m b_m) = a_{m+1}\frac{(b_{m+1} - b_m)}{\Delta_x} + b_m\frac{(a_{m+1} - a_m)}{\Delta_x} \\ &= a_{m+1}\tilde{\partial}_x b_m + b_m \tilde{\partial}_x a_m,\end{aligned} \tag{4}$$

and also that

$$\tilde{\partial}_x(a_m b_m) = a_m \tilde{\partial}_x b_m + b_{m+1}\tilde{\partial}_x a_m. \tag{5}$$

Similarly, we can show that

$$\begin{aligned}\hat{\partial}_x(a_m b_m) &= a_m \hat{\partial}_x b_m + b_{m-1}\hat{\partial}_x a_m \\ &= a_{m-1}\hat{\partial}_x b_m + b_m \hat{\partial}_x a_m.\end{aligned} \tag{6}$$

3. Discrete Vector Calculus

In continuum calculus, a vector calculus exists, which is the foundation for the description of electromagnetic theory. We shall show that an equivalent discrete vector calculus exists.

3.1. Discrete Gradient

In higher dimensions, a gradient can be defined. Consider a function $f(x, y)$ in two dimensions which we replace by f_{mn} at the point (m, n) as shown in Figure 1. A gradient operator in two dimensions can be defined such that

$$\begin{aligned}\tilde{\mathbf{g}}_{mn} = \tilde{\nabla} f_{mn} &= \hat{x}\tilde{\partial}_x f_{mn} + \hat{y}\tilde{\partial}_y f_{mn} \\ &= \hat{x}\frac{1}{\Delta_x}(f_{m+1,n} - f_{m,n}) + \hat{y}\frac{1}{\Delta_y}(f_{m,n+1} - f_{mn})\end{aligned} \tag{7}$$

or that

$$\tilde{\nabla} = \hat{x}\tilde{\partial}_x + \hat{y}\tilde{\partial}_y, \tag{7b}$$

where $\hat{x}$ and $\hat{y}$ are unit vectors in the $x-$ and $y-$ directions,respectively. From the definitions of $\tilde{\partial}_x$ and $\tilde{\partial}_y$ in the previous section, $\tilde{\partial}_x f_{mn}$ is defined at $(m+\frac{1}{2}, n)$, while $\tilde{\partial}_y f_{mn}$ is defined at $(m, n+\frac{1}{2})$. Therefore, the $\hat{x}$ and $\hat{y}$ components of $\tilde{\nabla} f_{mn}$ are defined on different grid points, or on the arrowheads as shown in Figure 1 even though $\tilde{\mathbf{g}}_{mn}$ is associated with the point (m, n). We call such a vector associated with the point (m, n) the *fore-vector* and denote it with a tilde.

Consequently, it can be shown that

$$\lim_{\Delta \to 0} \tilde{\nabla} f_{mn} = \nabla f(x, y), \tag{8}$$

where $\Delta \to 0$ implies that both Δ_x and $\Delta_y \to 0$. The error of the above equation is of $O(\Delta)$ where Δ is proportional to the discretization size. The definition of $\tilde{\nabla}$ can be extended to three dimensions as

$$\tilde{\mathbf{g}}_{mnp} = \tilde{\nabla} f_{mnp} = \hat{x}\tilde{\partial}_x f_{mnp} + \hat{y}\tilde{\partial}_y f_{mnp} + \hat{z}\tilde{\partial}_z f_{mnp}, \tag{9}$$

where $\tilde{\mathbf{g}}_{mnp}$ is a fore-vector associated with the point (m, n, p). Analogously, if we define

$$\hat{\nabla} = \hat{x}\hat{\partial}_x + \hat{y}\hat{\partial}_y + \hat{z}\hat{\partial}_z, \tag{10}$$

then

$$\hat{\mathbf{g}}_{mnp} = \hat{\nabla} f_{mnp} \tag{11}$$

is a vector with $\hat{x}$, $\hat{y}$, $\hat{z}$ components defined on $(m - \frac{1}{2}, n, p)$, $(m, n - \frac{1}{2}, p)$, and $(m, n, p - \frac{1}{2})$, respectively. We call such a vector the *back-vector* and denote it with a hat. It is quite clear that

$$\lim_{\Delta \to 0} \hat{\nabla} f_{mnp} = \nabla f(x, y, z). \tag{12}$$

with error of $O(\Delta)$.

3.2. Discrete Divergence

To define divergence, we have to define a vector field by values defined on the arrowheads as shown in Figure 1. Then in the vicinity of the point (m, n), we have field values $f^x_{m+\frac{1}{2},n}$, $f^x_{m-\frac{1}{2},n}$, $f^y_{m,n+\frac{1}{2}}$, and $f^y_{m,n-\frac{1}{2}}$. Hence, a fore-vector field can be associated with the point (m, n) as

$$\tilde{\mathbf{F}}_{mn} = \hat{x} f^x_{m+\frac{1}{2},n} + \hat{y} f^y_{m,n+\frac{1}{2}}. \tag{13}$$

Then, we define $\hat{\nabla} \cdot \tilde{\mathbf{F}}_{mn}$ associated with the point (m, n) as

$$d_{mn} = \hat{\nabla} \cdot \tilde{\mathbf{F}}_{mn} = \hat{\partial}_x f^x_{m+\frac{1}{2},n} + \hat{\partial}_y f^y_{m,n+\frac{1}{2}}. \tag{14}$$

It is clear that if $\mathbf{F}(x, y) = \lim\limits_{\Delta \to 0} \tilde{\mathbf{F}}_{mn}$,

$$\lim_{\Delta \to 0} \hat{\nabla} \cdot \tilde{\mathbf{F}}_{mn} = \nabla \cdot \mathbf{F}(x, y). \tag{15}$$

The error of the above equation is of $O(\Delta^2)$ because the above is equivalent to central differencing.

The above can be extended to three dimensions to yield

$$d_{mnp} = \hat{\nabla} \cdot \tilde{\mathbf{F}}_{mnp} = \hat{\partial}_x f^x_{m+\frac{1}{2},n,p} + \hat{\partial}_y f^y_{m,n+\frac{1}{2},p} + \hat{\partial}_z f^z_{m,n,p+\frac{1}{2}}. \tag{16}$$

The physical interpretation of the above is that if (m, n, p) denotes the center of a cube (see Figure 2), then $f^x_{m\pm\frac{1}{2},n,p}$, $f^y_{m,n\pm\frac{1}{2},p}$ and $f^z_{m,n,p\pm\frac{1}{2}}$ represent fluxes flowing through the faces of a cuboid of volume $\Delta_x\Delta_y\Delta_z$. Equation (16) denotes the differential fluxes through the six faces of the cuboid.

Similarly, if $\hat{\mathbf{F}}_{mnp}$ is a back-vector associated with the point (m,n,p), then $\tilde{\nabla}\cdot\hat{\mathbf{F}}_{mnp}$ gives the divergence at the point (m,n,p).

3.3. Discrete Curl

A curl is a measure of the circulation or rotation of a field about a point. In two dimensions, with vectors associated with the arrowheads as shown in Figure 1, then a curl can be associated with the solid-dot point as shown in Figure 3. Define a fore-vector field at (m,n), i.e., $\tilde{\mathbf{F}}_{mn}$ as in (13). Then, a curl can be associated with $(m+\frac{1}{2}, n+\frac{1}{2})$ such that

$$\begin{aligned}\hat{z}r_{m+\frac{1}{2},n+\frac{1}{2}} &= \tilde{\nabla}\times\tilde{\mathbf{F}}_{mn} = \hat{z}(\tilde{\partial}_x f^y_{m,n+\frac{1}{2}} - \tilde{\partial}_y f^x_{m+\frac{1}{2},n}) \\ &= \hat{z}\left[\frac{(f^y_{m+1,n+\frac{1}{2}} - f^y_{m,n+\frac{1}{2}})}{\Delta_x} + \frac{(f^x_{m+\frac{1}{2},n+1} - f^x_{m+\frac{1}{2},n})}{\Delta_y}\right]. \end{aligned} \tag{17}$$

It is clear that (17) is a measure of the circulation of the field around the solid dot in Figure 3. Also,

$$\lim_{\Delta\to 0}\tilde{\nabla}\times\tilde{\mathbf{F}}_{mn} = \nabla\times\mathbf{F}(x,y). \tag{18}$$

The above equation has an error of $O(\Delta^2)$ because it is equivalent to central differencing.

The above can be extended to three dimensions. For a field defined by

$$\tilde{\mathbf{F}}_{mnp} = \hat{x}f^x_{m+\frac{1}{2},n,p} + \hat{y}f^y_{m,n+\frac{1}{2},p} + \hat{z}f^z_{m,n,p+\frac{1}{2}}, \tag{19}$$

then

$$\begin{aligned}\tilde{\nabla}\times\tilde{\mathbf{F}}_{mnp} &= \hat{x}(\tilde{\partial}_y f^z_{m,n,p+\frac{1}{2}} - \tilde{\partial}_z f^y_{m,n+\frac{1}{2},p}) \\ &+ \hat{y}(\tilde{\partial}_z f^x_{m+\frac{1}{2},n,p} - \tilde{\partial}_x f^z_{m,n,p+\frac{1}{2}}) \\ &+ \hat{z}(\tilde{\partial}_x f^y_{m,n+\frac{1}{2},p} - \tilde{\partial}_y f^x_{m+\frac{1}{2},n,p}). \end{aligned} \tag{20}$$

To understand better the above physically, we let the field f be defined on the edges of a cube as shown in Figure 4.

Writing out (20) explicitly, we have

$$\begin{aligned}\tilde{\nabla}\times\tilde{\mathbf{F}}_{mnp} &= \hat{x}\left[\frac{(f^z_{m,n+1,p+\frac{1}{2}} - f^z_{m,n,p+\frac{1}{2}})}{\Delta_y} - \frac{(f^y_{m,n+\frac{1}{2},p+1} - f^y_{m,n+\frac{1}{2},p})}{\Delta_z}\right] \\ &+ \hat{y}\left[\frac{(f^x_{m+\frac{1}{2},n,p+1} - f^x_{m+\frac{1}{2},n,p})}{\Delta_z} - \frac{(f^z_{m+1,n,p+\frac{1}{2}} - f^z_{m,n,p+\frac{1}{2}})}{\Delta_x}\right] \\ &+ \hat{z}\left[\frac{(f^y_{m+1,n+\frac{1}{2},p} - f^y_{m,n+\frac{1}{2},p})}{\Delta_x} - \frac{(f^x_{m+\frac{1}{2},n+1,p} - f^x_{m+\frac{1}{2},n,p})}{\Delta_y}\right]. \end{aligned} \tag{21}$$

The above represents circulations of the field around the edges of the three faces of a cuboid shown in Figure 4. The $\hat{x}$-component in (21) is a measure of the circulation of the field along the loop OABC. The $\hat{y}$-component is a measure of the circulation of the field along the loop OCDE. Finally, the $\hat{z}$-component is a measure of the circulation along the loop OEFA. The above description is precisely that of the Yee algorithm [6].

We can associate $\tilde{\nabla} \times \tilde{\mathbf{F}}_{mnp}$ with a vector at the location $(m + \frac{1}{2}, n + \frac{1}{2}, p + \frac{1}{2})$, but the $\hat{x}$, $\hat{y}$, $\hat{z}$, components of the vector are actually defined at the dots as shown in Figure 4. We can call

$$\hat{\mathbf{B}}_{m+\frac{1}{2},n+\frac{1}{2},p+\frac{1}{2}} = \tilde{\nabla} \times \tilde{\mathbf{F}}_{mnp}. \tag{22}$$

Notice that $\hat{\mathbf{B}}_{m+\frac{1}{2},n+\frac{1}{2},p+\frac{1}{2}}$ is a back-vector associated with the point $(m + \frac{1}{2}, n + \frac{1}{2}, p + \frac{1}{2})$. Similarly, if we define $\hat{\mathbf{F}}_{mnp}$ to be a back-vector, then

$$\tilde{\mathbf{B}}_{m-\frac{1}{2},n-\frac{1}{2},p-\frac{1}{2}} = \hat{\nabla} \times \hat{\mathbf{F}}_{mnp} \tag{23}$$

is a fore-vector associated with the point $(m - \frac{1}{2}, n - \frac{1}{2}, p - \frac{1}{2})$.

3.4. Discrete Gauss' Theorem

We define the volume integral of $\hat{\nabla} \cdot \tilde{\mathbf{F}}_{mn}$ over a unit cell as shown in Figure 5 as

$$\hat{\nabla} \cdot \tilde{\mathbf{F}}_{mn}\Big|_{\text{cell}} \hat{d}V = (f^x_{m+\frac{1}{2},n} - f^x_{m-\frac{1}{2},n})\Delta_y + (f^y_{m,n+\frac{1}{2}} - f^y_{m,n-\frac{1}{2}})\Delta_x\Big|_{\text{cell}} \tag{24}$$

where $\hat{d}V = \Delta_x \Delta_y$ in two dimensions. Then the integration of $\hat{\nabla} \cdot \mathbf{F}$ over the volume V as shown in Figure 5 consisting of the sum of unit cells is

$$\int_V \hat{\nabla} \cdot \tilde{\mathbf{F}} \hat{d}V = \int_S \hat{n} \cdot \mathbf{f} \hat{d}S \tag{25}$$

where $\int_S \hat{n} \cdot \mathbf{f} \hat{d}S$ is a discrete summation over all of the fluxes leaving the surface S bounding V. Equation (25) is the discretized form of Gauss' theorem and is an exact relationship. It can be extended easily to three dimensions.

3.5. Discrete Stokes' Theorem

From the definition of curl, it is clear that on a unit cell,

$$\tilde{\nabla} \times \tilde{\mathbf{F}}_{mn}\Big|_{\text{cell}} \tilde{d}S = (f^x_{m+\frac{1}{2},n} - f^x_{m+\frac{1}{2},n+1})\Delta_x + (f^y_{m,n+\frac{1}{2}} - f^y_{m+1,n+\frac{1}{2}})\Delta_y\Big|_{\text{cell}} \tag{26}$$

where $\tilde{d}S = \Delta_x \Delta_y$ in two dimensions. It is quite clear that

$$\int_S \tilde{\nabla} \times \tilde{\mathbf{F}} \tilde{d}S = \int_C \hat{l} \cdot \tilde{\mathbf{F}} \tilde{d}l \tag{27}$$

where $\int_C \hat{l} \cdot \tilde{\mathbf{F}} \tilde{d}l$ is the discrete summation over all the circulations along C. The above is an exact relationship. Note that the unit cell in Figure 6 is shifted from that of Figure 5. Equation (27) can be generalized to three dimensions.

3.6. Discrete Laplacian Operator

With divergence and gradient defined, now we can define a Laplacian operator. Given a scalar field ϕ_{mn}, a field can be defined such as (7) so that

$$\tilde{\mathbf{g}}_{mn} = \tilde{\nabla}\phi_{mn} = \hat{x}\tilde{\partial}_x\phi_{mn} + \hat{y}\tilde{\partial}_y\phi_{mn}. \tag{28}$$

Then $\hat{\nabla} \cdot \tilde{\mathbf{g}}_{mn}$ is

$$\hat{\nabla} \cdot \tilde{\nabla}\phi_{mn} = \hat{\partial}_x\tilde{\partial}_x\phi_{mn} + \hat{\partial}_y\tilde{\partial}_y\phi_{mn}. \tag{29}$$

It is clear that

$$\lim_{\Delta\to 0} \hat{\nabla} \cdot \tilde{\nabla}\phi_{mn} = \nabla^2\phi(x, y). \tag{30}$$

A forward difference followed by a backward difference is equivalent to central differencing. The above equation is accurate to $O(\Delta^2)$. Consequently, we can write Poisson's equation as

$$\hat{\nabla} \cdot \tilde{\nabla}\phi_{mn} = -\rho_{mn}/\epsilon_0 \tag{31}$$

and the wave equation as

$$\hat{\nabla} \cdot \tilde{\nabla}\phi_{mn} + k_{mn}^2\phi_{mn} = \rho_{mn}/\epsilon_0. \tag{32}$$

The above can be extended to three dimensions. From this point onward, we can define a discrete Laplacian, $\tilde{\nabla}^2 = \hat{\nabla} \cdot \tilde{\nabla}$. Since $\tilde{\partial}_x\hat{\partial}_x = \hat{\partial}_x\tilde{\partial}_x$, we have $\tilde{\nabla} \cdot \hat{\nabla} = \hat{\nabla} \cdot \tilde{\nabla}$.

3.7. Discrete Green's Theorem

In continuum vector calculus, we have Green's theorem, which is the foundation for Huygens' principle. The continuum theory states that

$$\iint_V (\phi\nabla^2 g - g\nabla^2\phi)dV = \int_S (\phi\nabla g - g\nabla\phi) \cdot \hat{n}dS. \tag{33}$$

The above is a consequence of Gauss' theorem applied to the following identity, i.e.,

$$\nabla \cdot (\phi\nabla g - g\nabla\phi) = \phi\nabla^2 g - g\nabla^2\phi, \tag{34}$$

which in turn is a consequence of the identity

$$\nabla \cdot (\phi\nabla g) = \phi\nabla^2 g + \nabla\phi \cdot \nabla g. \tag{35}$$

We like to examine if (33) to (35) hold true in a discrete space.

First, we examine the one-dimensional analog of (35) or

$$\partial_x(\phi\partial_x g) = \phi\partial_x^2 g + (\partial_x\phi)(\partial_x g). \tag{36}$$

The discretized analog of (36) using (6) and (3) is

$$\hat{\partial}_x(\phi_m\tilde{\partial}_x g_m) = \phi_m\hat{\partial}_x\tilde{\partial}_x g_m + (\hat{\partial}_x\phi_m)(\hat{\partial}_x g_m). \tag{37}$$

Extending (37) to two dimensions, we have

$$\begin{aligned}\hat{\nabla} \cdot (\phi_{mn}\tilde{\nabla}g_{mn}) &= \hat{\partial}_x(\phi_{mn}\tilde{\partial}_x g_{mn}) + \hat{\partial}_y(\phi_{mn}\tilde{\partial}_y g_{mn}). \\ &= \phi_{mn}\hat{\partial}_x\tilde{\partial}_x g_{mn} + (\hat{\partial}_x\phi_{mn})(\hat{\partial}_x g_{mn}) \\ &+ \phi_{mn}\hat{\partial}_y\tilde{\partial}_y g_{mn} + (\hat{\partial}_y\phi_{mn})(\hat{\partial}_y g_{mn}),\end{aligned} \tag{38}$$

or

$$\hat{\nabla} \cdot (\phi_{mn} \tilde{\nabla} g_{mn}) = \phi_{mn} \hat{\nabla} \cdot \tilde{\nabla} g_{mn} + (\hat{\nabla} \phi_{mn}) \cdot (\hat{\nabla} g_{mn}). \tag{39}$$

Using (39), it becomes clear that

$$\hat{\nabla} \cdot (\phi_{mn} \tilde{\nabla} g_{mn} - g_{mn} \tilde{\nabla} \phi_{mn}) = \phi_{mn} \tilde{\nabla}^2 g_{mn} - g_{mn} \tilde{\nabla}^2 \phi_{mn}. \tag{40}$$

Using the discrete Gauss' theorem, we have the discrete Green's theorem

$$\int_V (\phi_{mn} \tilde{\nabla}^2 g_{mn} - g_{mn} \tilde{\nabla}^2 \phi_{mn}) \hat{d}V = \int_S (\phi_{mn} \tilde{\nabla} g_{mn} - g_{mn} \tilde{\nabla} \phi_{mn}) \cdot \hat{n} \hat{d}S. \tag{41}$$

The above is an exact relationship, and is easily extended to three dimensions.

3.8. Discrete Huygens' Principle

Consider a Green's function that satisfies the equation

$$(\tilde{\nabla}^2 + k_{\mathbf{m}}^2) \tilde{g}_{\mathbf{m},\mathbf{m}'} = -\delta_{\mathbf{m}-\mathbf{m}'} \tag{42}$$

where $\mathbf{m} = (m, n)$ in two dimensions and $\mathbf{m} = (m, n, p)$ in three dimensions. $\delta_{\mathbf{m}-\mathbf{m}'} = 1/\hat{d}V$ when $\mathbf{m} = \mathbf{m}'$; $\delta_{\mathbf{m}-\mathbf{m}'} = 0$ when $\mathbf{m} \neq \mathbf{m}'$; and $\hat{d}V = \Delta_x \Delta_y \Delta_z$. Next, we assume that

$$(\tilde{\nabla}^2 + k_{\mathbf{m}}^2) \phi_{\mathbf{m}} = 0. \tag{43}$$

Multiplying (42) by $\phi_{\mathbf{m}}$ and (43) by $g_{\mathbf{m},\mathbf{m}'}$ and discretely integrate over $\mathbf{m}$, we have

$$\int_V (\phi_{\mathbf{m}} \tilde{\nabla}^2 \tilde{g}_{\mathbf{m},\mathbf{m}'} - \tilde{g}_{\mathbf{m},\mathbf{m}'} \tilde{\nabla}^2 \phi_{\mathbf{m}}) \hat{d}V = -\phi_{\mathbf{m}'}. \tag{44}$$

By the discrete Green's theorem, we have

$$\int_{S+S_{inf}} (\phi_{\mathbf{m}} \tilde{\nabla} \tilde{g}_{\mathbf{m},\mathbf{m}'} - \tilde{g}_{\mathbf{m},\mathbf{m}'} \tilde{\nabla} \phi_{\mathbf{m}}) \cdot \hat{n} \hat{d}S = \phi_{\mathbf{m}'} \tag{45}$$

where V is bounded by S and S_{inf}. The integration over S_{inf} vanishes by the radiation condition. Hence, we have Huygens' principle in a discrete space which is

$$\phi_{\mathbf{m}} = \int_S (\phi_{\mathbf{m}'} \tilde{\nabla} \tilde{g}_{\mathbf{m}',\mathbf{m}} - \tilde{g}_{\mathbf{m}',\mathbf{m}} \tilde{\nabla} \phi_{\mathbf{m}'}) \cdot \hat{n} \hat{d}S'. \tag{46}$$

The above is an exact relationship.

4. Discrete Electromagnetic Theory

If we define a discrete time variable as $t = l\Delta t$, then we can express $\phi_{\mathbf{m}}(t)$ as $\phi_{\mathbf{m}}^l$. An analog of ∂_t becomes $\tilde{\partial}_t$, and ∂_t^2 becomes $\hat{\partial}_t \tilde{\partial}_t$. In this manner, discrete Maxwell's equations can be written as

$$\tilde{\nabla} \times \tilde{\mathbf{E}}_{\mathbf{m}}^l = -\tilde{\partial}_t \hat{\mathbf{B}}_{m+\frac{1}{2}}^{l-\frac{1}{2}}, \tag{47a}$$

$$\hat{\nabla} \times \hat{\mathbf{H}}_{\mathbf{m}+\frac{1}{2}}^{l-\frac{1}{2}} = \hat{\partial}_t \tilde{\mathbf{D}}_{\mathbf{m}}^l + \tilde{\mathbf{J}}_{\mathbf{m}}^{l-\frac{1}{2}}, \tag{47b}$$

$$\tilde{\nabla} \cdot \hat{\mathbf{B}}_{\mathbf{m}+\frac{1}{2}}^{l-\frac{1}{2}} = 0, \tag{47c}$$

$$\hat{\nabla} \cdot \tilde{\mathbf{D}}_{\mathbf{m}}^l = \rho_{\mathbf{m}}^l, \tag{47d}$$

where $\mathbf{m}+\frac{1}{2}$ is an abbreviation for $(m+\frac{1}{2}, n+\frac{1}{2}, p+\frac{1}{2})$. Notice that (47c) and (47d) follow by taking the divergence of (47a) and (47b) and the use of the discretized form of the charge continuity equation $\hat{\nabla}\cdot\tilde{\mathbf{J}}+\hat{\partial}_t\rho=0$. Hence, the discrete Maxwell's equations are consistent with charge conservation, the same case as the continuum Maxwell's equations. The fact that the discrete Maxwell's equations preserve the charge conservation law makes it attractive for numerical simulation, because no spurious solution is generated.

It is to be noted that other ways of discretizing Maxwell's equations may not conserve charge. For instance, in the finite element method (FEM), a naive implementation gives rise to fictitious charges and spurious solutions , because the basis elements used to represent the electric and magnetic flux may not be divergence free. Hence, special care has to taken to ensure that the basis elements are divergence free.

In the above, $\hat{\mathbf{B}}$ and $\hat{\mathbf{H}}$ are back-vectors, while $\tilde{\mathbf{D}}$, $\tilde{\mathbf{E}}$, and $\tilde{\mathbf{J}}$ are fore-vectors. The above is Yee's algorithm [6] written in a form analogous to the familiar Maxwell's equations. Similar notations have also been introduced in [8]. If the constitutive relations are included, i.e.,

$$\hat{\mathbf{B}}_{\mathbf{m}+\frac{1}{2}} = \overline{\boldsymbol{\mu}}_{\mathbf{m}+\frac{1}{2}} \cdot \hat{\mathbf{H}}_{\mathbf{m}+\frac{1}{2}}, \tag{48a}$$

$$\tilde{\mathbf{D}}_{\mathbf{m}} = \overline{\boldsymbol{\epsilon}}_{\mathbf{m}} \cdot \tilde{\mathbf{E}}_{\mathbf{m}}, \tag{48b}$$

where $\overline{\boldsymbol{\mu}}$ and $\overline{\boldsymbol{\epsilon}}$ are permeability and permittivity tensors, respectively, the above equations can be solved numerically on a digital computer by time-matching. Also, Equations (47) are accurate to $O(\Delta^2)$ because only central differencing is involved.

Notice that the choice of $\mathbf{B}$ and $\mathbf{H}$ as back-vectors and $\mathbf{D}$, $\mathbf{E}$, and $\mathbf{J}$ as fore-vectors is arbitrary. One could have rewritten (47) with the converse. Also, a time harmonic form of (47) can be easily written.

4.1. Discrete Vector Wave Equation

The vector wave equation can be derived as follows:

$$\hat{\nabla}\times\overline{\boldsymbol{\mu}}_{\mathbf{m}+\frac{1}{2}}^{-1}\cdot\tilde{\nabla}\times\tilde{\mathbf{E}}_{\mathbf{m}}^{l}+\tilde{\partial}_t\hat{\partial}_t\overline{\boldsymbol{\epsilon}}_m\cdot\tilde{\mathbf{E}}_{\mathbf{m}}^{l}=\tilde{\partial}_t\tilde{\mathbf{J}}_{\mathbf{m}}^{l-\frac{1}{2}}. \tag{49}$$

If we let $\mathbf{E}_{\mathbf{m}}^{l}=\mathbf{E}_{\mathbf{m}}e^{-i\omega l\Delta t}$, representing a time harmonic field, then

$$\tilde{\partial}_t \Rightarrow \frac{1}{\Delta t}(e^{-i\omega\Delta t}-1)=\frac{-2i}{\Delta t}e^{-i\omega\frac{\Delta t}{2}}\sin\left(\frac{\omega\Delta t}{2}\right)=-i\Omega e^{-i\delta_t}, \tag{50a}$$

$$\hat{\partial}_t \Rightarrow \frac{1}{\Delta t}(1-e^{i\omega\Delta t})=\frac{-2i}{\Delta t}e^{-i\omega\frac{\Delta t}{2}}\sin\left(\frac{\omega\Delta t}{2}\right)=-i\Omega e^{i\delta_t}, \tag{50b}$$

$$\tilde{\partial}_t\hat{\partial}_t \Rightarrow -\Omega^2=-4\sin^2\left(\frac{\omega\Delta t}{2}\right)/(\Delta t)^2, \tag{50c}$$

where $\delta_t=\omega\Delta t/2$. Hence, for a time harmonic solution, the above becomes

$$\hat{\nabla}\times\overline{\boldsymbol{\mu}}_{\mathbf{m}+\frac{1}{2}}^{-1}\cdot\tilde{\nabla}\times\tilde{\mathbf{E}}_{\mathbf{m}}-\Omega^2\overline{\boldsymbol{\epsilon}}_{\mathbf{m}}\cdot\tilde{\mathbf{E}}_{\mathbf{m}}=i\Omega\tilde{\mathbf{J}}_{\mathbf{m}}. \tag{51}$$

Similarly, we can derive

$$\tilde{\nabla}\times\overline{\boldsymbol{\epsilon}}_{\mathbf{m}}^{-1}\cdot\hat{\nabla}\times\hat{\mathbf{H}}_{\mathbf{m}+\frac{1}{2}}-\Omega^2\overline{\boldsymbol{\mu}}_{\mathbf{m}+\frac{1}{2}}\cdot\hat{\mathbf{H}}_{\mathbf{m}+\frac{1}{2}}=0. \tag{52}$$

In the above, $\tilde{\mathbf{J}}_{\mathbf{m}}^{l-\frac{1}{2}} = \tilde{\mathbf{J}}_{\mathbf{m}} e^{-i\omega(l-\frac{1}{2})\Delta t}$, and $\hat{\mathbf{H}}_{\mathbf{m}+\frac{1}{2}}^{l-\frac{1}{2}} = \hat{\mathbf{H}}_{\mathbf{m}+\frac{1}{2}} e^{-i\omega(l-\frac{1}{2})\Delta t}$. Notice that Ω is real even when $\Delta t \neq 0$. When $\Delta t \to 0$, $\Omega \to \omega$.

4.2. Discrete Scalar-Wave Equation and Plane-Wave Solution

For a homogeneous, isotropic and source-free medium, $\overline{\boldsymbol{\mu}}$ and $\overline{\boldsymbol{\epsilon}}$ are scalar constants. Hence (51) becomes

$$\hat{\nabla} \times \tilde{\nabla} \times \tilde{\mathbf{E}}_{\mathbf{m}} - \Omega^2 \mu \epsilon \tilde{\mathbf{E}}_{\mathbf{m}} = 0. \tag{53}$$

It can be shown that

$$\hat{\nabla} \times (\tilde{\nabla} \times \tilde{\mathbf{E}}_{\mathbf{m}}) = \tilde{\nabla}(\hat{\nabla} \cdot \tilde{\mathbf{E}}_{\mathbf{m}}) - \tilde{\nabla}^2 \tilde{\mathbf{E}}_{\mathbf{m}} \tag{54}$$

where $\tilde{\nabla}^2 = \hat{\nabla} \cdot \tilde{\nabla}$. Since $\hat{\nabla} \cdot \tilde{\mathbf{E}}_{\mathbf{m}} = 0$, we have

$$\tilde{\nabla}^2 \tilde{\mathbf{E}}_{\mathbf{m}} + K^2 \tilde{\mathbf{E}}_{\mathbf{m}} = 0$$

where $K^2 = \Omega^2 \mu \epsilon$. The above is equivalent to three scalar-wave equations each of which is

$$(\tilde{\nabla}^2 + K^2)\phi_{\mathbf{m}} = 0. \tag{55}$$

If we let

$$\phi_{\mathbf{m}} = A e^{ik_x m\Delta_x + ik_y n\Delta_y + ik_z p\Delta_z}, \tag{56}$$

we have

$$\tilde{\partial}_x \Rightarrow \frac{1}{\Delta_x}(e^{ik_x\Delta_x} - 1) = \frac{2i}{\Delta_x} e^{ik_x\frac{\Delta_x}{2}} \sin\left(\frac{k_x\Delta_x}{2}\right) = iK_x e^{i\delta_x}, \tag{57a}$$

$$\hat{\partial}_x \Rightarrow \frac{1}{\Delta_x}(1 - e^{-ik_x\Delta_x}) = \frac{2i}{\Delta_x} e^{-ik_x\frac{\Delta_x}{2}} \sin\left(\frac{k_x\Delta_x}{2}\right) = iK_x e^{-i\delta_x}, \tag{57b}$$

$$\hat{\partial}_x\tilde{\partial}_x \Rightarrow -K_x^2 = -4\sin^2\left(\frac{k_x\Delta_x}{2}\right)/\Delta_x^2, \tag{57c}$$

where $\delta_x = k_x\Delta_x/2$. Similar relations can be derived for $\tilde{\partial}_y$, $\hat{\partial}_y$, $\tilde{\partial}_z$, and $\hat{\partial}_z$. Therefore,

$$\tilde{\nabla}^2\phi_{\mathbf{m}} = -(K_x^2 + K_y^2 + K_z^2)\phi_{\mathbf{m}}. \tag{58}$$

In order for (56) to satisfy (55) , we need

$$K_x^2 + K_y^2 + K_z^2 = \Omega^2/c^2, \tag{59}$$

where $c = 1/\sqrt{\mu\epsilon}$. The above equation is the dispersion relation that relates k_x, k_y, and k_z with ω. It can be shown that when k_x, k_y and k_z are real, ω is always real only if $c\Delta t < \Delta$, where $\Delta = \left(\frac{1}{\Delta_x^2} + \frac{1}{\Delta_y^2} + \frac{1}{\Delta_z^2}\right)^{-\frac{1}{2}}$. Hence, a plane wave can propagate without attenuation or growth on the lattice if such a criterion is satisfied. This criterion is also related to causality [14]. Since $\hat{\nabla} \cdot \tilde{\mathbf{E}}_{\mathbf{m}} = 0$, the plane wave solution must satisfy

$$\mathbf{K} \cdot \tilde{\mathbf{E}}_{\mathbf{m}} = 0, \tag{60}$$

where $\mathbf{K} = \hat{x}K_x e^{-i\delta_x} + \hat{y}K_y^{-i\delta_y} + \hat{z}K_z e^{-i\delta_z}$.

4.3. Discrete Poynting's theorem

It is not clear how Poynting's theorem should be like on a regular lattice. Before deriving Poynting's theorem, a Poynting vector has to be defined. In the discrete space, the $\tilde{\mathbf{E}}$ field, a fore-vector, is defined at $\mathbf{m}$, while the $\hat{\mathbf{H}}$ field, a back-vector, is defined at $\mathbf{m}+\frac{1}{2}$. A special cross-product has to be defined so that it has the conservative property of Poynting's theorem.

A vector that has this property is

$$\tilde{\mathbf{S}}_{\mathbf{m}} = \tilde{\mathbf{E}}_{\mathbf{m}} \otimes \hat{\mathbf{H}}^*_{\mathbf{m}+\frac{1}{2}}$$

where the above cross-product is defined as

$$\begin{aligned}\tilde{\mathbf{E}}_{\mathbf{m}} \otimes \hat{\mathbf{H}}^*_{\mathbf{m}+\frac{1}{2}} &= \hat{x}[(\mathcal{Z}_x\tilde{E}_{y,\mathbf{m}})\hat{H}^*_{z,\mathbf{m}+\frac{1}{2}} - (\mathcal{Z}_x\tilde{E}_{z,\mathbf{m}})\hat{H}^*_{y,\mathbf{m}+\frac{1}{2}}] \\ &+ \hat{y}[(\mathcal{Z}_y\tilde{E}_{z,\mathbf{m}})\hat{H}^*_{x,\mathbf{m}+\frac{1}{2}} - (\mathcal{Z}_y\tilde{E}_{x,\mathbf{m}})\hat{H}^*_{z,\mathbf{m}+\frac{1}{2}}] \\ &+ \hat{z}[(\mathcal{Z}_z\tilde{E}_{x,\mathbf{m}})\hat{H}^*_{y,\mathbf{m}+\frac{1}{2}} - (\mathcal{Z}_z\tilde{E}_{y,\mathbf{m}})\hat{H}^*_{x,\mathbf{m}+\frac{1}{2}}]. \end{aligned} \tag{61}$$

In the above, $\mathcal{Z}_x$, $\mathcal{Z}_y$, and $\mathcal{Z}_z$ are shifting operators having the properties that

$$\mathcal{Z}_x f_{m,n,p} = f_{m+1,n,p}, \tag{62a}$$
$$\mathcal{Z}_y f_{m,n,p} = f_{m,n+1,p}, \tag{62b}$$
$$\mathcal{Z}_z f_{m,n,p} = f_{m,n,p+1}. \tag{62c}$$

Taking the divergence of $\tilde{\mathbf{S}}$, we have

$$\begin{aligned}\hat{\nabla}\cdot\tilde{\mathbf{S}}_{\mathbf{m}} &= \hat{\partial}_x[(\mathcal{Z}_x\tilde{E}_{y,\mathbf{m}})\hat{H}^*_{z,\mathbf{m}+\frac{1}{2}} - (\mathcal{Z}_x\tilde{E}_{z,\mathbf{m}})\hat{H}^*_{y,\mathbf{m}+\frac{1}{2}}] \\ &+ \hat{\partial}_y[(\mathcal{Z}_y\tilde{E}_{z,\mathbf{m}})\hat{H}^*_{x,\mathbf{m}+\frac{1}{2}} - (\mathcal{Z}_y\tilde{E}_{x,\mathbf{m}})\hat{H}^*_{z,\mathbf{m}+\frac{1}{2}}] \\ &+ \hat{\partial}_z[(\mathcal{Z}_z\tilde{E}_{x,\mathbf{m}})\hat{H}^*_{y,\mathbf{m}+\frac{1}{2}} - (\mathcal{Z}_z\tilde{E}_{y,\mathbf{m}})\hat{H}^*_{x,\mathbf{m}+\frac{1}{2}}]. \end{aligned} \tag{63}$$

Looking at the first term in (63) and using (6), we have

$$\begin{aligned}&\hat{\partial}_x[(\mathcal{Z}_x\tilde{E}_{y,\mathbf{m}})\hat{H}^*_{z,\mathbf{m}+\frac{1}{2}} - (\mathcal{Z}_x\tilde{E}_{z,\mathbf{m}})\hat{H}^*_{y,\mathbf{m}+\frac{1}{2}}] \\ &\qquad = \tilde{E}_{y,\mathbf{m}}\hat{\partial}_x\hat{H}^*_{z,\mathbf{m}+\frac{1}{2}} + \hat{H}^*_{z,\mathbf{m}+\frac{1}{2}}\tilde{\partial}_x\tilde{E}_{y,\mathbf{m}} \\ &\qquad\quad - \tilde{E}_{z,\mathbf{m}}\hat{\partial}_x\hat{H}^*_{y,\mathbf{m}+\frac{1}{2}} - \hat{H}^*_{y,\mathbf{m}+\frac{1}{2}}\tilde{\partial}_x\tilde{E}_{z,\mathbf{m}}. \end{aligned} \tag{64}$$

Writing out all terms in (63) explicitly, we deduce that

$$\hat{\nabla}\cdot(\tilde{\mathbf{E}}_{\mathbf{m}} \otimes \hat{\mathbf{H}}^*_{\mathbf{m}+\frac{1}{2}}) = \hat{\mathbf{H}}^*_{\mathbf{m}+\frac{1}{2}}\cdot\tilde{\nabla}\times\tilde{\mathbf{E}}_{\mathbf{m}} - \tilde{\mathbf{E}}_{\mathbf{m}}\cdot\hat{\nabla}\times\hat{\mathbf{H}}^*_{\mathbf{m}+\frac{1}{2}}. \tag{65}$$

Using the time-harmonic form of (47), we have

$$\hat{\nabla}\cdot\tilde{\mathbf{S}}_{\mathbf{m}} = i\Omega\hat{\mathbf{H}}^*_{\mathbf{m}+\frac{1}{2}}\cdot\hat{\mathbf{B}}_{\mathbf{m}+\frac{1}{2}} - i\Omega\tilde{\mathbf{E}}_{\mathbf{m}}\cdot\tilde{\mathbf{D}}^*_{\mathbf{m}} - \tilde{\mathbf{E}}_{\mathbf{m}}\cdot\tilde{\mathbf{J}}_{\mathbf{m}}. \tag{66}$$

Equation (66) is Poynting's theorem in the discrete space. It is an exact relationship. An integral form can be derived easily. Following the case of the continuum Poynting's theorem, a medium is lossless if $\overline{\boldsymbol{\mu}} = \overline{\boldsymbol{\mu}}^{\dagger}$ and $\overline{\boldsymbol{\epsilon}} = \overline{\boldsymbol{\epsilon}}^{\dagger}$ [13, 14].

4.4. Reciprocity Theorem

Reciprocity is an important property of electromagnetic fields in a continuum space displayed by media when $\overline{\boldsymbol{\epsilon}}$ and $\overline{\boldsymbol{\mu}}$ are symmetric tensors [13]. The reciprocal nature of electromagnetic fields is related to $\nabla \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla\times$, which is a symmetric operator [14].

For a discrete space, reciprocity would require that $\hat{\nabla} \times \overline{\boldsymbol{\mu}}^{-1}_{\mathbf{m}+\frac{1}{2}} \cdot \tilde{\nabla}\times$ be a symmetric operator. To show this, we need to show that [14]

$$\langle \tilde{\mathbf{E}}_1, \hat{\nabla} \times \overline{\boldsymbol{\mu}}^{-1} \cdot \tilde{\nabla} \times \tilde{\mathbf{E}}_2 \rangle = \langle \tilde{\mathbf{E}}_2, \hat{\nabla} \times \overline{\boldsymbol{\mu}}^{-1} \cdot \tilde{\nabla} \times \tilde{\mathbf{E}}_1 \rangle \tag{67}$$

where

$$\langle \tilde{\mathbf{E}}_1, \hat{\nabla} \times \overline{\boldsymbol{\mu}}^{-1} \cdot \tilde{\nabla} \times \tilde{\mathbf{E}}_2 \rangle = \int_V \tilde{\mathbf{E}}_1 \cdot \hat{\nabla} \times \overline{\boldsymbol{\mu}}^{-1} \cdot \tilde{\nabla} \times \tilde{\mathbf{E}}_2 \hat{d}V. \tag{68}$$

In the above, the subscripts $\mathbf{m}$ and $\mathbf{m}+\frac{1}{2}$ are implicitly implied. To prove (67), we use the identity (65) to deduce that

$$\hat{\nabla} \cdot (\tilde{\mathbf{E}}_1 \otimes \overline{\boldsymbol{\mu}}^{-1} \cdot \tilde{\nabla} \times \tilde{\mathbf{E}}_2) = \tilde{\nabla} \times \tilde{\mathbf{E}}_1 \cdot \overline{\boldsymbol{\mu}}^{-1} \cdot \tilde{\nabla} \times \tilde{\mathbf{E}}_2 - \tilde{\mathbf{E}}_1 \cdot \hat{\nabla} \times \overline{\boldsymbol{\mu}}^{-1} \cdot \tilde{\nabla} \times \tilde{\mathbf{E}}_2. \tag{69}$$

Integrating (69) over V and letting S be the surface bounding V, we have, after using the discrete Gauss' theorem,

$$\langle \tilde{\mathbf{E}}_1, \hat{\nabla} \times \overline{\boldsymbol{\mu}}^{-1} \cdot \tilde{\nabla} \times \tilde{\mathbf{E}}_2 \rangle = \langle \tilde{\nabla} \times \tilde{\mathbf{E}}_1, \overline{\boldsymbol{\mu}}^{-1} \cdot \tilde{\nabla} \times \tilde{\mathbf{E}}_2 \rangle - \mu^{-1} \int_S \hat{n} \cdot (\tilde{\mathbf{E}}_1 \otimes \tilde{\nabla} \times \tilde{\mathbf{E}}_2) \hat{d}S. \tag{70}$$

We assume that at S, $\overline{\boldsymbol{\mu}} = \mu$, a constant.

The first term on the right-hand side of (70) is symmetric about the interchange of $\tilde{\mathbf{E}}_1$ and $\tilde{\mathbf{E}}_2$ if $\overline{\boldsymbol{\mu}}$ is symmetric. Suitable boundary conditions can be chosen on S to make the second term vanish or be symmetric, and (67) will be satisfied for a bounded region. For an unbounded region, if we can show that the second term of (70) is symmetric about $\tilde{\mathbf{E}}_1$ and $\tilde{\mathbf{E}}_2$, when $S \to S_{inf}$, then the left-hand side of (70) would be symmetric about $\tilde{\mathbf{E}}_1$ and $\tilde{\mathbf{E}}_2$ and (67) would be satisfied. To this end, we write

$$\begin{aligned} A = \int_{S_{inf}} \hat{n} \cdot (\tilde{\mathbf{E}}_1 \otimes \tilde{\nabla} \times \tilde{\mathbf{E}}_2) \hat{d}S &= \int_V \hat{\nabla} \cdot (\tilde{\mathbf{E}}_1 \otimes \tilde{\nabla} \times \tilde{\mathbf{E}}_2) \hat{d}V \\ &= \int_V (\tilde{\nabla} \times \tilde{\mathbf{E}}_1) \cdot (\tilde{\nabla} \times \tilde{\mathbf{E}}_2) \hat{d}V - \int_V (\tilde{\mathbf{E}}_1 \cdot \hat{\nabla} \times \tilde{\nabla} \times \tilde{\mathbf{E}}_2) \hat{d}V. \end{aligned} \tag{71}$$

A in (71) would be symmetric if the last term in (71) is symmetric. To show this, we let

$$\int_V (\tilde{\mathbf{E}}_1 \cdot \hat{\nabla} \times \tilde{\nabla} \times \tilde{\mathbf{E}}_2) \hat{d}V = \int_V \tilde{\mathbf{E}}_1 \cdot \tilde{\nabla}\hat{\nabla} \cdot \tilde{\mathbf{E}}_2 \hat{d}V - \int_V \tilde{\mathbf{E}}_1 \cdot \tilde{\nabla}^2 \tilde{\mathbf{E}}_2 \hat{d}V. \tag{72}$$

Using indicial notation, the integrand of the first term on the right-hand side of (72), with the help of Equation (5), is

$$
\begin{aligned}
\tilde{\mathbf{E}}_1 \cdot \tilde{\nabla}\hat{\nabla} \cdot \tilde{\mathbf{E}}_2 &= \tilde{E}_{1i}\tilde{\partial}_i\hat{\partial}_j\tilde{E}_{2j} \\
&= \tilde{\partial}_i(\tilde{E}_{1i}\hat{\partial}_j\tilde{E}_{2j}) - (\tilde{\partial}_i\tilde{E}_{1i})(\mathcal{Z}_i\hat{\partial}_j\tilde{E}_{2j}) \\
&= \hat{\partial}_i[\mathcal{Z}_i(\tilde{E}_{1i}\hat{\partial}_j\tilde{E}_{2j})] - (\tilde{\partial}_i\tilde{E}_{1i})(\mathcal{Z}_i\hat{\partial}_j\tilde{E}_{2j}) \qquad (73)
\end{aligned}
$$

where i, j can be either x, y, z, and $\mathcal{Z}_i$ is a shifting operator in the i-th direction. When we integrate over the volume V, we obtain

$$
\int\limits_V \tilde{\mathbf{E}}_1 \cdot \tilde{\nabla}\hat{\nabla} \cdot \tilde{\mathbf{E}}_2 \hat{d}V = \int\limits_{S_{inf}} \hat{n} \cdot [\overline{\mathbf{Z}} \cdot (\tilde{\mathbf{E}}_1\hat{\nabla} \cdot \tilde{\mathbf{E}}_2)]\hat{d}S - \int\limits_V (\tilde{\partial}_i\tilde{E}_{1i})(\mathcal{Z}_i\hat{\partial}_j\tilde{E}_{2j})\hat{d}V. \qquad (74)
$$

On S_{inf}, $\hat{\nabla} \cdot \tilde{\mathbf{E}}_2 = 0$ so that the first term on the right-hand side is zero. The second term, if V is an infinite volume, is the same as

$$
\begin{aligned}
\lim_{V\to\infty} \int (\tilde{\partial}_i\tilde{E}_{1i})(\mathcal{Z}_i\hat{\partial}_j\tilde{E}_{2j})\hat{d}V &= \lim_{V\to\infty} \int\limits_V \hat{\partial}_i\tilde{E}_{1i}\hat{\partial}_j\tilde{E}_{2j}\hat{d}V \\
&= \int\limits_{V_{inf}} (\hat{\nabla} \cdot \tilde{\mathbf{E}}_1)(\hat{\nabla} \cdot \tilde{\mathbf{E}}_2)\hat{d}V. \qquad (75)
\end{aligned}
$$

Therefore, we conclude that

$$
\int\limits_{V_{inf}} \tilde{\mathbf{E}}_1 \cdot \tilde{\nabla}\hat{\nabla} \cdot \tilde{\mathbf{E}}_2 \hat{d}V = \int\limits_{V_{inf}} (\hat{\nabla} \cdot \tilde{\mathbf{E}}_1)(\hat{\nabla} \cdot \tilde{\mathbf{E}}_2)\hat{d}V \qquad (76)
$$

which is symmetric about $\tilde{\mathbf{E}}_1$ and $\tilde{\mathbf{E}}_2$.

The last term in (72) is symmetric as it is the same as

$$
\begin{aligned}
\int\limits_V \tilde{E}_{1i}\tilde{\partial}_j\hat{\partial}_j\tilde{E}_{2i}\hat{d}V &= \int\limits_V \tilde{\partial}_j(\tilde{E}_{1i}\hat{\partial}_j\tilde{E}_{2i})\hat{d}V - \int\limits_V \hat{\partial}_j\tilde{E}_{1i}\hat{\partial} j\tilde{E}_{2i}\hat{d}V \\
&= \int\limits_S \hat{n} \cdot (\tilde{E}_{1i}\hat{\nabla}\tilde{E}_{2i})\hat{d}S - \int\limits_V \hat{\nabla}\tilde{E}_{1i} \cdot \hat{\nabla}\tilde{E}_{2i}\hat{d}V. \qquad (77)
\end{aligned}
$$

When $S \to S_{inf}$, all waves become plane waves and $\tilde{\nabla}\tilde{E}_{2i} = \mathbf{K}\tilde{E}_{2i}$, where $\mathbf{K}$ is as defined in (60). Then the right-hand side of (77) is obviously symmetric about $\tilde{\mathbf{E}}_1$ and $\tilde{\mathbf{E}}_2$. As a result, we have proved (67) .

Consequently, for a source $\tilde{\mathbf{J}}_1$ generating $\tilde{\mathbf{E}}_1$ and $\tilde{\mathbf{J}}_2$ generating $\tilde{\mathbf{E}}_2$ via the following equations.

$$
\hat{\nabla} \times \overline{\boldsymbol{\mu}}^{-1} \cdot \tilde{\nabla} \times \tilde{\mathbf{E}}_1 - \Omega^2\overline{\boldsymbol{\epsilon}} \cdot \tilde{\mathbf{E}}_1 = i\Omega\tilde{\mathbf{J}}_1 \qquad (78)
$$

$$
\hat{\nabla} \times \overline{\boldsymbol{\mu}}^{-1}\tilde{\nabla} \times \tilde{\mathbf{E}}_2 - \Omega^2\overline{\boldsymbol{\epsilon}} \cdot \tilde{\mathbf{E}}_2 = i\Omega\tilde{\mathbf{J}}_2. \qquad (79)
$$

If $\overline{\boldsymbol{\mu}}$ and $\overline{\boldsymbol{\epsilon}}$ are symmetric tensors, then

$$\langle \tilde{\mathbf{E}}_2, \tilde{\mathbf{J}}_1 \rangle = \langle \tilde{\mathbf{E}}_1, \tilde{\mathbf{J}}_2 \rangle \tag{80}$$

which is the reciprocity theorem in the discrete space. It is an exact relationship. The scalar-wave version can be easily derived as a special case of the above.

5. Conclusions

We have shown that by using discrete calculus, electromagnetic theory can be written for a rectilinear lattice. Moreover, theorems of electromagnetic theory in the continuum world are preserved on this lattice. Even though we have not presented it here, it is straight forward to show that the uniqueness theorem is preserved in the discretized world. Hence, discrete electromagnetic theory has an important place in the description of observed physical electromagnetic phenomena if the discretization length scale is chosen small enough to match all observed electromagnetic phenomena. This discretization length scale is on the order of the wavelength of gamma rays.

Discrete electromagnetic theory, with a larger discretization length scale, however, can also be used for the finite difference computer simulation of the electromagnetic field [15]. The preservation of the theorems of electromagnetics in the discrete world implies that these theorems can be used to check the correctness of the numerical simulations. The self-consistent and self-contained nature of a discrete electromagnetic theory places such numerical simulation on a firmer theoretical foundation.

In the finite difference world, Equations (47c) and (47d) are directly derivable from Equations (47a) and (47b), via the use of the charge continuity equation in the discretized world. This also implies that Equations (47) are charge conserving in the discretized world. Hence, there is no existence of spurious charges giving rise to spurious solution. However, a straightforward discretization of Maxwell's equations using the finite element method would usually yield spurious solution, and finite element method has to be used with care to discretize Maxwell's equations [12]. In using the language of differential forms to discretized electromagnetic field [12], the electric field and magnetic field are discretized as one forms, while the electric flux and magnetic flux are discretized as two forms. This generally leads to inconsistencies on a tetrahedral mesh when constitutive relations are defined to relate the field to the flux, which is to relate a one form to a two form. In a rectilinear mesh in the finite difference world, the field and the flux are both masquerading as a one form and a two form simultaneously, because on the primary grid, the field appears as a one form, but on the dual grid, the field appears as a two form. Because of this, there is no problem of defining a constitutive relation for the discretized Maxwell's equations on a rectilinear grid.

However, a completely regular lattice is a convenient but not a general description of the discrete world. This work can be extended to the case of a discrete electromagnetic theory on an irregular lattice. However, it will be a subject of another paper.

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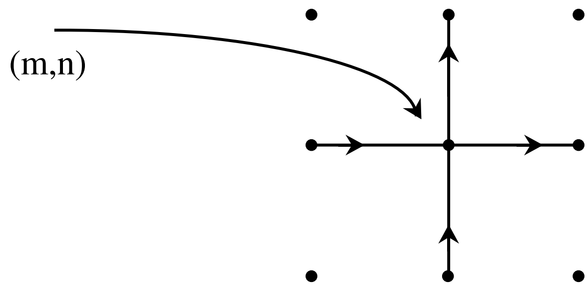

Figure 1. Gradient associated with the point (m,n) *as given by Equation (7).*

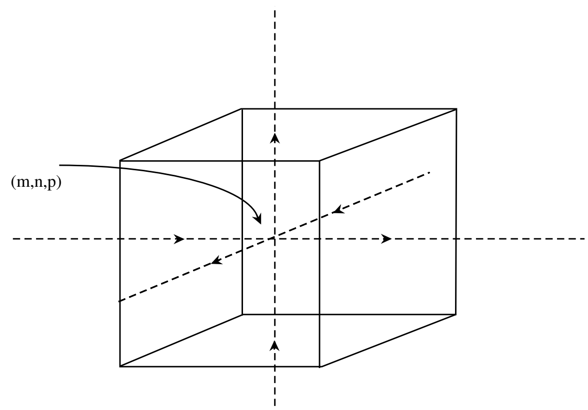

Figure 2. Geometry for the definition of discrete divergence in three dimensions.

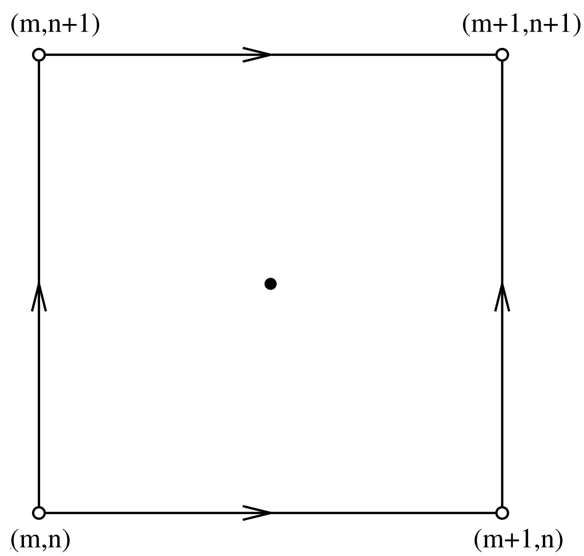

Figure 3. The definition of discrete curl in two dimensions.

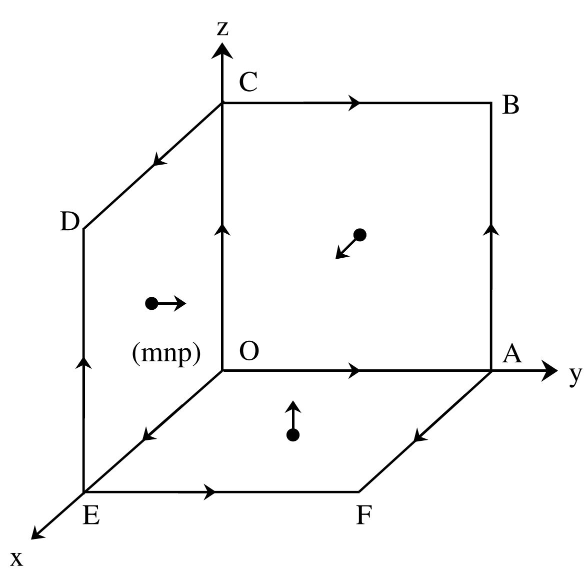

Figure 4. The definition of discrete curl in three dimensions.

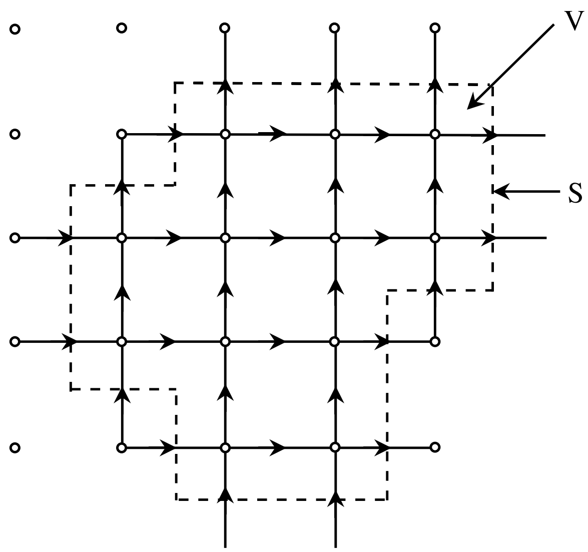

Figure 5. Definition of discrete Gauss' theorem in two dimensions.

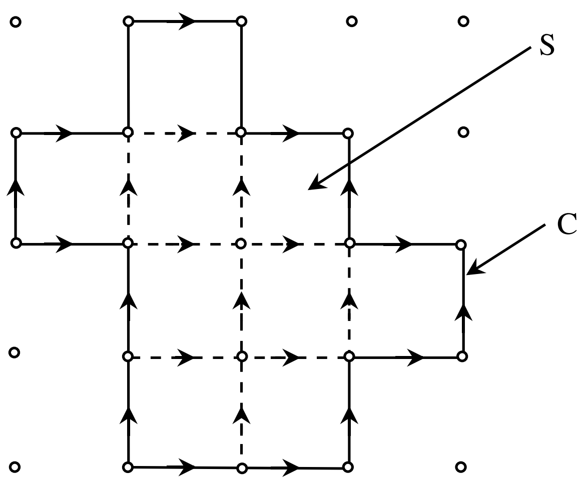

Figure 6. Definition of discrete Stokes' theorem in three dimensions.