Fast Solution Methods in Electromagnetics †

(Invited)

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Abstract

Various methods for efficiently solving electromagnetic problems are presented. Electromagnetic scattering problems can be roughly classified into surface and volume problems, while fast methods are either differential or integral equation based. The resultant systems of linear equations are either solved directly or iteratively. A review of various differential equation solvers, their complexities, and memory requirements is given. The issues of grid dispersion and hybridization with integral equation solvers are discussed. Several fast integral equation solvers for surface and volume scatterers are presented. These solvers have reduced computational complexities and memory requirements.

1. Introduction

Computational electromagnetics is a fascinating discipline that has drawn the attention of mathematicians, engineers, physicists, and computer scientists alike. It is a discipline that creates a symbiotic marriage between mathematics, physics, computer science, and various application fields. Computational techniques for solving electromagnetic wave scattering problems involving large complex bodies and for analyzing wave propagation through inhomogeneous media have been intensely studied by many researchers in the past [1-12]. This is due to the importance of this research in many practical applications, such as the prediction of the Radar Cross Section (RCS) of complex objects like aircraft, the interaction of antenna elements with aircraft and ships, the environmental effects of vegetation, clouds, and aerosols on electromagnetic wave propagation, the interaction of electromagnetic waves with biological media, and the propagation of signals in high-speed and millimeter-wave circuits.

Due to the large electrical dimensions of typical aircraft, past efforts to ascertain their scattering cross section and their interaction with antennas have exploited approximate high frequency techniques such as the shooting and bouncing ray method [13]. However, the recent phenomenal growth in computer technology, coupled with the development of fast algorithms with reduced computational complexity and memory requirements, have made a rigorous numerical solution of the problem of scattering from electrically large objects feasible. These numerical techniques involve either solving partial-differential equations with the Finite-Difference Method (FDM) [6-9] or the Finite-Element Method (FEM) [10-12] which result in sparse matrices, or integral equations which are converted into dense matrix equations using the Method Of Moments (MOM) [1-5].

In a previous paper [14], we underscored the importance of reducing the computational complexity of computational electromagnetics techniques, especially for large-scale electromagnetic problems. We reviewed several direct solvers with reduced computational complexity whereby the solution is sought for all right-hand sides. These direct solvers are the Recursive Aggregate T-Matrix Algorithm (RATMA) [15,16], and the Nested Equivalence Principle Algorithm (NEPAL) [16,17]. In this paper, we will first review differential equation solvers, and discuss their computational complexity. We next focus on recent work in integral

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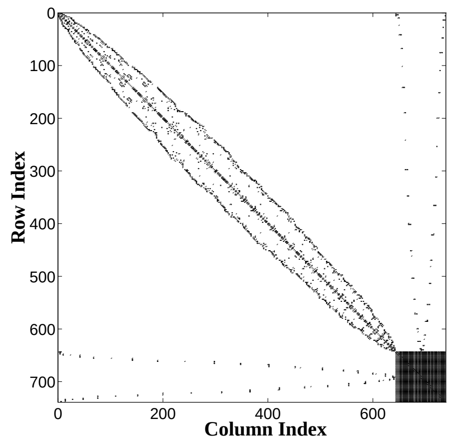

Figure 1. *Proper ordering of the elements of an FEM matrix that is coupled to a surface integral equation causes the dense matrix to reside at the bottom right-hand corner, making the system matrix suitable for the matrix-partitioning method.*

equation solvers, and contrast their complexities with those of differential equation solvers. Throughout, we will focus primarily on iterative solvers, which are used ubiquitously for solving both differential and integral equations. Iterative solvers, in general, require less memory storage, and exhibit reduced computational complexities when compared to direct solvers. Hence, they portend an important method for large scale computing.

2. Differential Equation Solvers

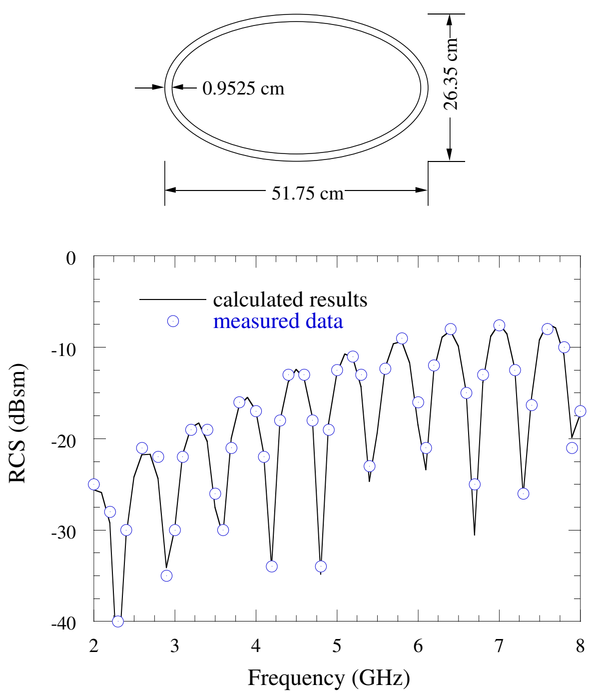

Figure 2. *Backscatter RCS of an elliptic contoured crack in a ground plane as a function of frequency at the incidence angle $\theta = 70^o$ (20^o from grazing incidence) and $\phi = 85^o$ (5^o from the the minor axis). The crack is 0.635 cm deep and filled with air.*

A popular way to solve electromagnetic problems is to solve the associated partial differential equation directly. These methods can be considered as the first fast solution methods in electromagnetics because one can solve an N unknown problem with computational complexity less than $O(N^3)$ and memory requirement less than $O(N^2)$. Differential equation solvers usually involve either the finite element method (FEM) [10-12] or the finite difference method (FDM) [6-9]. The pertinent matrix equation is sparse with $O(N)$ nonzero elements. Consequently, a matrix-vector multiply can be performed in $O(N)$ operations. By properly ordering the elements, the bandwidth of the pertinent matrix equation can be compressed and inverted very efficiently [18]. Differential equation solvers are usually applied to volumetric problems, and hence, the following discussion is pertinent to volumetric cases.

Partial differential equations (PDEs) for electromagnetics can be roughly categorized into elliptic type (static or Laplacian-like) for low frequencies, hyperbolic type (wavelike) for high frequencies, and parabolic type (diffusionlike) for intermediate frequencies and lossy media. Elliptic PDEs have the advantage of positive definiteness [18]; hence, when iterative methods are used to solve the associated matrix equation, a definitive statement can be made about their convergence rates. For instance, when the conjugate gradient (CG) method [19,20] is used to solve the Poisson equation, it converges in $O(N^{0.5})$ steps in two dimensions, and in $O(N^{0.33})$ in three dimensions. When the multigrid method is used to solve the same equation, the number of iterations is independent of the size of the problem [18]. As a consequence, the total computational labor associated with the conjugate gradient method to solve such problems scales as $N^{1.5}$ in 2D and $N^{1.33}$ in 3D, while it scales as N for multigrid methods.

For hyperbolic (wavelike) problems which are indefinite, these computational complexities can only be regarded as lower bounds, because with the change of geometry, resonance can occur, and the number of iterations needed for convergence in an iterative solver can diverge. Multigrid solvers exploit the scale-invariant nature of an elliptic (Laplacian-like) equation to reduce the computational complexity. But when applied to the Helmholtz wave equation (hyperbolic), the computational complexity is not reduced, because the Helmholtz wave equation is not scale invariant.

When the finite difference time domain (FDTD) method is used to solve the wave equation directly in the time domain, the computational complexity is the same as CG ($N^{1.5}$ in 2D and $N^{1.33}$ in 3D) where they are lower bounds [21] except that FDTD generates the solution for all time, and hence all frequencies at once. It is also an optimal algorithm in the sense that it generates $O(N^\alpha)$ numbers in $O(N^\alpha)$ operations.

Of interest also is the spectral Lanczos decomposition method (SLDM) [22,23]. While it does not reduce the computational complexity compared to CG, it offers an advantage when there are large regions of homogeneity. Also, it can generate the solution for all frequencies without additional computational cost [23]. For numerical simulation of waveguides where a large section of uniformity exists, the method of lines [24] and the numerical mode matching method [25,26] offer an advantage over other differential equation methods in terms of speed.

When applied to a scattering problem, a PDE solver requires absorbing boundary conditions (ABCs) [21] to truncate the simulation region. Many ABCs have been proposed so that the sparsity of the matrix can be maintained. However, these ABCs are approximate and have to be imposed at a substantial distance from the scatterer to reduce the errors incurred by them. Recently, an absorbing material boundary condition (AMBC), called perfectly matched layer (PML), has been suggested by Berenger [27] and worked on intensely by a number of workers [28-35]. This AMBC is particularly well-suited for the parallel implementation of FDTD solvers because it permits parallel computers to operate in a single-instruction-multiple-data (SIMD) mode [28].

Another approach to truncate the simulation region is to employ the eigenfunction expansion of the scattered field outside a separable boundary. This separable boundary can either be circular or elliptical in 2D and spherical or spheroidal in 3D. The resultant method is often referred to as the unimoment method [36-39]. Through the use of coupled basis functions for the separable boundary, the differential equation can be effectively decoupled from the dense matrix generated from the eigenfunction expansion. The dimension of this dense matrix is, however, rather small. In 2D, it is about kd where d is the largest linear dimension of the scatterer.

Alternatively, surface integral equations (which can be considered to be numerically exact ABCs) can be used to truncate the mesh of the differential equation solvers [40,41]. By so doing, the boundary of the simulation region can be brought much closer to the surface of the scatterer, thereby reducing the size of the simulation region and the number of associated unknowns. However, such a method of "absorbing" the outgoing wave results in a partially dense matrix in the final matrix system for the problem.

By a proper ordering of the nodes in FEM [41,42], the dense matrix will reside only at the bottom right-hand corner of the matrix system as shown in Figure 1. In this manner, the inverse of the matrix system can be found by the matrix-partitioning method. When nested-dissection ordering [43] is applied to the sparse part, and LU decomposition is applied to the dense part, the overall computational complexity is of $O(N^{1.5})$ in 2D, and of $O(N^2)$ in 3D. The memory requirements are $O(N \log N)$ in 2D and $O(N^{4/3})$ in 3D [18].

When iterative methods are used to solve the matrix system as shown in Figure 1, the matrix-vector multiply from the dense submatrix could become a bottleneck in 3D or for thinly coated metallic scatterers. However, with the use of fast integral equation solvers [44,45], this bottleneck could be removed. The example of hybridizing a fast integral equation solver and FEM has been illustrated [46]. Figure 2 shows the comparison of such a calculation with experiments [47] when applied to an elliptically contoured crack in a ground plane.

One of the drawbacks of differential equation solvers is the grid dispersion error incurred [48-50]. The grid dispersion error causes a wave to have a different phase velocity on a grid compared to the exact solution. This error can be suppressed by using a higher grid density, but at the expense of increased computational labor. Because the error is cumulative, it is particularly pronounced for simulation over a large region or for large scatterers. To suppress the error, the grid density has to increase as the simulation region increases in size. For second-order accurate schemes, the grid density in one dimension (number of points per wavelength) has to grow as $(kd)^{0.5}$ where d is the "diameter" of the simulation region and k is the wavenumber of the wave [49]. Therefore, the number of unknowns grows as $(kd)^{1.5}$ in one dimension. Hence, in two dimensions, the number of unknowns scales as $(kd)^3$ while in three dimensions, it scales as $(kd)^{4.5}$. A remedy for this is to use a higher-order accurate differential equation solver [51,52] or to couple the differential equation solver to an integral equation solver when large homogeneous regions exist.

3. Integral Equation Solvers

Alternatively, a scattering problem can be cast into an integral equation. Integral equation solvers usually involve a smaller number of unknowns than differential equation solvers because only the induced sources are unknowns, whereas in a differential equation, the field is the unknown. For example, for a metallic scatterer, the induced current resides only on the surface of the scatterer. Hence, for a scatterer in a 3D space, the induced current exists in a space of smaller dimensions, greatly reducing the number of unknowns required to accurately represent the solution, except for inhomogeneous scatterers. However, integral equation solvers result in dense matrices. If the matrix equation is then solved by LU decomposition (Gaussian elimination) or alternatively by an iterative technique such as the conjugate gradient (CG) or related methods [19,20], the computational labor may be excessive. LU decomposition requires $O(N^3)$ operations and $O(N^2)$ memory storage and provides a solution for all excitations of the scatterer. CG requires $O(N^2)$ operations per iteration for dense matrices, because the most costly step in a CG iteration is in the matrix-vector multiply. In general, the number of iterations grows with the electrical size of the scatterer. A straightforward implementation of CG requires $O(N^2)$ memory storage, providing a solution that is valid for only one excitation. However, it is possible to iteratively solve the the pertinent equation concurrently for multiple right-hand sides, thereby exploiting as much as possible the redundancies in the right-hand sides [53].

The high computational complexity of the aforementioned solution schemes precludes their application to the analysis of scattering from large structures. Many researchers have attempted to reduce the complexity of the traditional MOM algorithm by reducing the computational labor of the pertinent matrix-vector multiplies. For surface scatterers, Rokhlin [44] proposed the Fast-Multipole Method (FMM) to reduce the computational complexity of matrix-vector multiplies in an iterative method. Canning [54] has developed the impedance-matrix localization the (IML) method, which uses basis functions that produce directed beams. This results in a sparse MOM matrix, which in turn, expedites a matrix-vector multiply. The IML works well for smooth surfaces, but not for non-smooth surfaces. In a similar spirit, the complex multipole beam

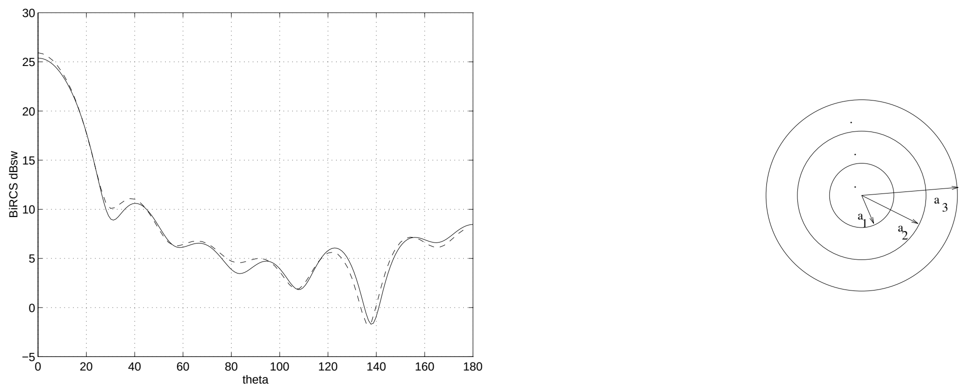

Figure 3. *The bistatic RCS of a three-layed spherical scatterer. The solid line is from the Mie series solution; the dashed line is for the numerical solution using BiCG-FFT.* $a_1 = 0.169m$, $\epsilon_{r,1} = 1.2$, $a_2 = 0.339m$, $\epsilon_{r,2} = 2.0$, *and* $a_3 = 0.508m$, $\epsilon_{r,3} = 2.4$; *with* $31 \times 31 \times 31$ *grids. This problem has about 90,000 unknowns and the frequency is 590 MHz* (λ_0*=0.508 m)*

approach has been introduced [55], but it again works only for smooth surfaces.

Wavelet transforms [56-61] have also been used to yield sparse matrices that can be solved rapidly. Since wavelets are scale invariant, they are well-suited for solving static or low frequency (elliptic) problems. When wavelets are used to sparsify matrices resulting from an integral equation of static, they sparsify the matrices to $O(N \log N)$ elements, reducing the operation count of a matrix-vector multiply to $O(N \log N)$. For wavelike problems, even though these methods expedite matrix-vector multiplies, they do not reduce the computational complexity [61] when the scatterer size grows with respect to wavelength. Many methods have been proposed in the past which, even though will reduce solution time, do not reduce the computational complexity [62-65].

For volumetric scatterers, several recursive and nesting algorithms have been developed to directly obtain the solutions of integral equations for all right-hand sides [14-17]. Also, in an iterative method, the FFT can be used to expedite the matrix-vector multiply and reduce the computational complexity and memory requirement for solving such scattering problems [66-77].

Here, we will first discuss fast methods to solve volume integral equations rapidly using FFT [66-72]. Then for surface integral equations, we will first discuss the use of wavelet transforms to expedite matrix-vector multiplies in an iteration solution method. Finally, we will discuss the use of the FMM, related methods, and various multilevel algorithms to accelerate matrix-vector multiplies in an iterative solver.

4. Iterative Methods for Volume Scattering

Scattering from a volumetric object can be analyzed efficiently using iterative methods where the bottleneck is the matrix-vector multiply involving a dense matrix. The matrix-vector multiply represents the action of the Green's operator on induced currents in the scatterer. Since the Green's operator is translationally invariant, this action can be written as a convolutional integral:

$$\int_V d\mathbf{r}' g(\mathbf{r} - \mathbf{r}') j(\mathbf{r}'), \tag{1}$$

where $g(\mathbf{r})$ is the Green's function, and $j(\mathbf{r})$ is the induced current. Such action can be expedited using an FFT, with a complexity of $O(N \log N)$ [66-70]. However, in 3D, the Green's function for electromagnetic scattering is highly singular (as in the dyadic Green's function). Therefore, a high sampling rate is needed

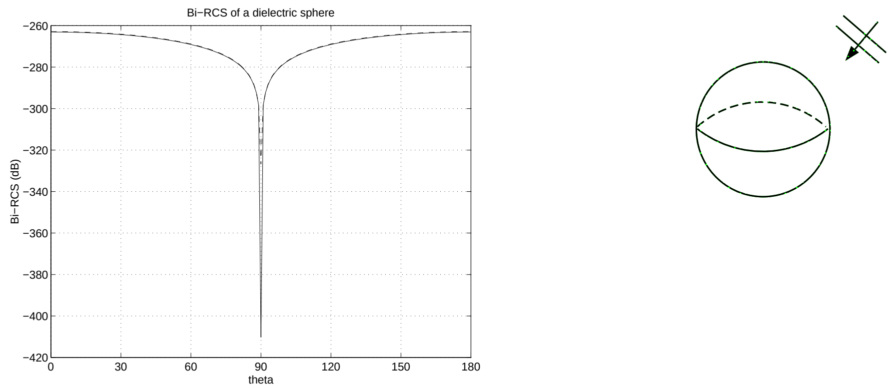

Figure 4. *The bistatic RCS of a dielectric sphere computed using the BiCG-FFT T matrix method. The solid line is from the Mie series solution; the dashed line is the numerical solution. Here, the radius* $= 10^{-5}\lambda_0$ *and* $\epsilon_r = 4.0$. *A* $16 \times 16 \times 16$ *grid is used. The number of iterations needed to solve this problem is independent of the number of unknowns at such a low frequency.*

to perform the above convolution accurately. To mitigate the singularity of the dyadic Green's function, a difference operator is used to approximate the differential operator in the dyadic Green's function in [67]. In a similar vein, [68] proposed the use of a weak formulation of the integral equation plus a spherical mean approximation. In [69], the induced current is expanded in terms of a continuous function, even though the induced current, which is proportional to $(\epsilon - \epsilon_0)\mathbf{E}$, should be a discontinuous function.

Alternatively, we can discretize the above integral by projecting it onto a smaller subspace using the MOM [1], converting the integral operator into a matrix operator. The singularity of the Green's operator is being mitigated by this projection. When the subspace is spanned by subdomain basis functions, and the mesh used is rectilinear, the pertinent matrix is (block) Toeplitz [70]. Consequently, the matrix-vector multiply can be performed exactly by using a FFT requiring $O(N \log N)$ operations. Figure 3 shows the bistatic RCS of a layered sphere computed using such a method. The sphere is modelled by 90,000 unknowns, and a matrix-vector multiply can be performed in several minutes on a 10 MFLOPS machine.

Alternatively, we can decompose the inhomogeneous scatterer into N subscatterers, whose scattering is characterized by a T matrix. Then a set of linear algebraic equations accounting for the multiple scattering between the subscatterers is derived. When the scatterers reside on a regular array, the pertinent matrix equation has a Toeplitz structure, and the FFT can be used to compute the matrix-vector multiply in $O(N \log N)$ operations [71,72].

Both this method and the MOM method avoid the singularity of the Green's function, and only a low sampling rate is needed to perform the FFT accurately. Figure 4 shows the bistatic RCS of a dielectric sphere computed using such a method. This method does not have low-frequency instability problems as opposed to some FEM formulations as discussed in [78,79].

Comparison of the efficiency of the CG-FFT method and RATMA has been presented in [73,74]. When a scatterer is lossless, RATMA is superior to CG-FFT. But when the scatterer is lossy, the number of iterations required is small, and CG-FFT is more efficient than RATMA.

The CG-FFT method can also be used to expedite the solution of the scattering from a cluster of randomly distributed spheres and randomly distributed cylinders. When the subscatterers do not reside on a regular array, a precorrected method can be used to derive a Toeplitz matrix structure, and FFT can again be used to accelerate the matrix-vector multiply [71,76]. The precorrected FFT method has also been used in the adaptive integral method (AIM) [77], which will be discussed in greater detail in Section 11.

5. Wavelets

There have been many attempts at using wavelets to solve scattering problems [56-61]. Such approaches have met with some success at lower frequencies due to the elliptic nature of the electrostatic problem. For instance, wavelets can be used to sparsify the boundary integral equation of electrostatics. The originally dense matrix resulting from discretizing this integral equation reaches a sparsity of $O(N \log N)$ after applying a wavelet transform [80]. This sparsity occurs because the integral operator belongs to the class of Calderon-Zygmund operators [80,81] where the kernel is infinitely smooth. A physical explanation is that, at low frequencies, wavelet function currents generate only localized fields. In other words, in electrostatics, the interaction between wavelet sources is mainly short range. This is particularly so for interactions due to the "fine features" of the sources. In addition, electrostatic problems are scale invariant, as are the wavelet bases.

For PDEs, the associated matrix is already sparse. Hence, there is no apparent advantage to applying a wavelet transform to such a matrix. However, for elliptic PDEs (static), the wavelet transform generates a matrix that can be easily preconditioned so that the resultant condition number of the matrix is of order one, irrespective of the size of the scatterer [82]. As a result, when an iterative solver is used, the number of iterations is independent of the problem size and it can be solved in $O(N)$ operations. Therefore, wavelets, for elliptic PDEs, offer advantages similar to those of multigrid.

Unfortunately, for wavelike problems, the associated integral equation has an oscillatory kernel. In other words, wavelike problems are not scale-invariant. Hence, there is no clear advantage to using a wavelet transform on such a kernel, as one can show that the sparsity of the matrix cannot be reduced to less than $O(N^2)$, the lower bound being related to the Nyquist sampling rate in Fourier analysis [61]. The physical explanation is that when the length scale of a wavelet equals or exceeds the wavelength, it becomes an efficient radiator. Hence, strong long-range interactions exist between these basis functions irrespective of the problem size. The long range interaction in electrodynamics falls off as $1/r$; this decay cannot be ignored even over large distances.

Local cosine transforms have been suggested as a remedy to this problem [83]. Local cosine current functions radiate a field that has a sharply directed beam pattern as in IML [54]. As a result, the matrix becomes sparse when the scatterer has a smooth surface. However, when the surface is rough, the local cosine current function loses its sharply directed beam pattern, and the matrix loses its sparsity.

Given a matrix equation resulting from discretizing an integral equation using the method of moments with a pulse basis,

$$\overline{\mathbf{A}} \cdot \mathbf{x} = \mathbf{b}, \tag{2}$$

the corresponding wavelet basis representation can be related to the pulse basis representation by a matrix transform

$$\mathbf{x} = \overline{\mathbf{U}} \cdot \mathbf{w}, \tag{3}$$

where $\overline{\mathbf{U}}$ is unitary when the wavelet basis is orthonormal (though nonorthogonal wavelets are also used). By using (3) in (2), we have

$$\overline{\mathbf{U}}^t \cdot \overline{\mathbf{A}} \cdot \overline{\mathbf{U}} \cdot \mathbf{w} = \overline{\mathbf{U}}^t \cdot \mathbf{b}, \tag{4}$$

or

$$\widetilde{\overline{\mathbf{A}}} \cdot \mathbf{w} = \widetilde{\mathbf{b}}, \tag{5}$$

where

$$\widetilde{\overline{\mathbf{A}}} = \overline{\mathbf{U}}^t \cdot \overline{\mathbf{A}} \cdot \overline{\mathbf{U}}, \tag{6a}$$

$$\widetilde{\mathbf{b}} = \overline{\mathbf{U}}^t \cdot \mathbf{b}. \tag{6b}$$

The matrix $\widetilde{\overline{\mathbf{A}}}$ is the moment method matrix represented in the wavelet basis. Figure 5 [61] shows two matrices from a 2D electrodynamic boundary integral equation for a circular scatterer and an L-shaped scatterer after wavelet transform using Daubechies wavelets [84]. It is seen that the bottom right-hand corner of the matrix remains dense.

Figure 6 [61] shows the matrix sparsity as a function of the number of unknowns for the circular scatterer and the L-shaped scatterer. It is clear that the fraction of nonzero elements remains a constant after the scatterer has increased to a certain size. Here, a discretization density of 10 points per wavelength is used throughout the study. Hence, as the size of the scatterer increases, its dimension increases with respect to the wavelength. However, if we keep the size of the scatterer constant with respect to the wavelength and increase the discretization density to increase the number of unknowns, then the sparsity of the matrix will

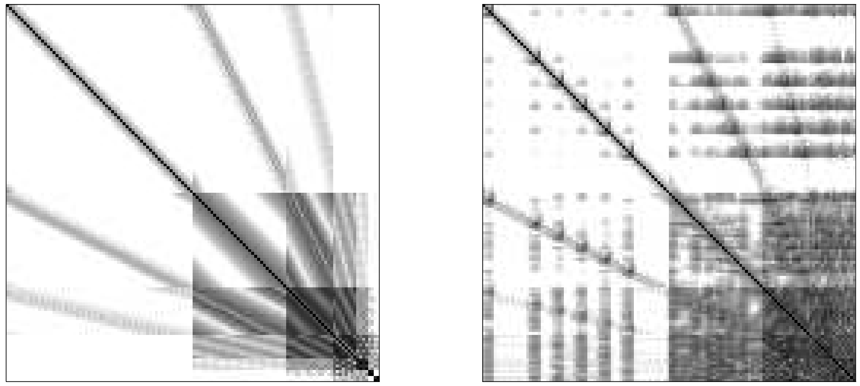

Figure 5. *The matrices after wavelet transform appear sparse but are dense at the bottom right-hand corner. The left one is for a circular cylinder, while the right one is for an L-shaped cylinder. Daubechies wavelets with eight vanishing moments are used.*

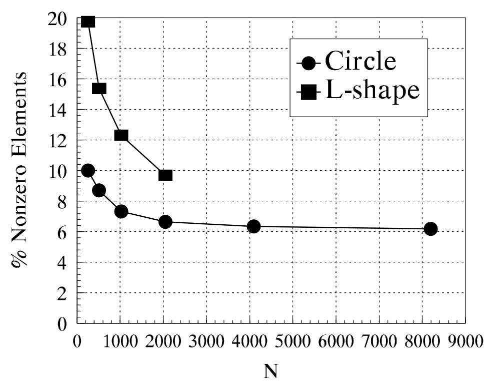

Figure 6. *The percentage of nonzero elements as a function of the number of unknowns. It is seen that the percentage does not go down after a while, because of the long-range interaction of wavelets in electrodynamics.*

increase as expected. The wavelet transform removes over-sampling of the unknowns beyond the Nyquist rate.

6. Fast Multipole Method

For surface structures, there exists no direct solver with reduced computational for efficiently solving the integral equation of scattering. Therefore, one resorts to an iterative solver whereby the computational complexity of a matrix-vector multiply can be reduced. Many methods for expediting matrix-vector multiplies have been proposed, but the FMM and its variants [44,45,85-92] hold most promise in providing a fast method that applies to scatterers of arbitrary geometry. The detailed mathematical description of the FMM for electromagnetic problems can be found in the aforementioned references. Therefore, we will describe this method from a heuristic viewpoint.

A matrix-vector multiply involving a dense matrix and a dense vector requires N^2 operations. This is illustrated by Figure 7. In essence, every element of a vector communicates with every other element

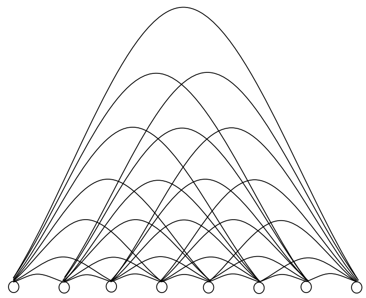

Figure 7. *A one-level matrix-vector multiply where all current elements talk directly to each other. The number of "links" is proportional to N^2 where N is the number of current elements.*

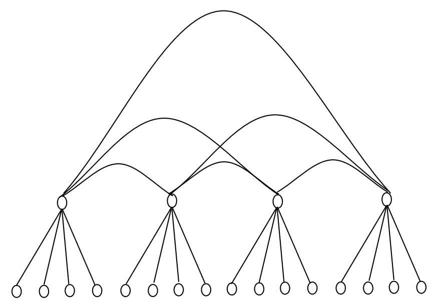

Figure 8. *A two-level matrix-vector multiply where "hubs" are established to reduce the number of direct "links" between the current elements. This could potentially reduce the complexity of a matrix-vector multiply.*

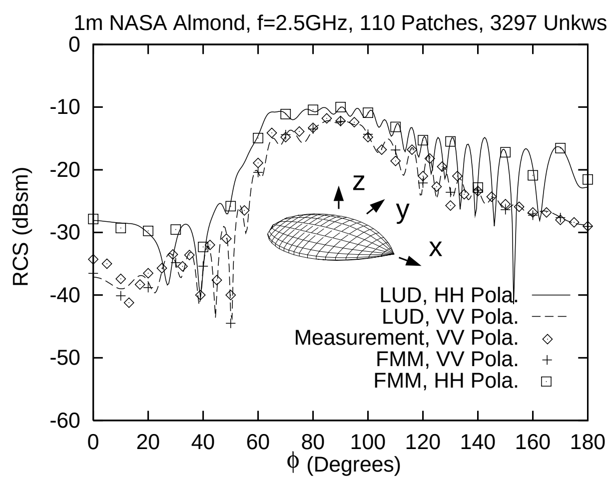

Figure 9. *Monostatic RCS of a one-meter long NASA almond at 2.5 GHz in the xy plane with $\theta = 90^o$. Five unknowns are used per wavelength. The results are computed with LU decomposition, and partially with FMM. The experimental measurement by Ohio State University [3] is also given for comparison.*

directly. Clearly, N^2 operations are needed. The above is like connecting N cities with direct flight routes. The number of flight routes increases as N^2. However, if "hubs" are introduced in the flight routes, then their number can be reduced, as shown in Figure 8, where the number of flight routes becomes less than N^2. Since Figure 8 represents a two-level structure, a matrix-vector multiply would have to be effected in three stages. Therefore, a matrix element has to be factored as a product of three terms. In other words, a matrix-vector multiply can be expressed as

$$\sum_{j=1}^{N} A_{ij} x_j = \mathbf{V}_{il}^t \cdot \sum_{l'=1}^{N/M} \overline{\boldsymbol{\alpha}}_{ll'} \cdot \sum_{j \epsilon \mathcal{G}_{l'}} \mathbf{V}_{l'j} x_j \quad \begin{cases} i \in \mathcal{G}_l, \\ l = 1, \cdots, N/M. \end{cases} \tag{7}$$

In the above, we assume that the N elements in the vector are divided in groups with M elements each. Therefore, there are a total of N/M groups $\mathcal{G}_l$. Moreover, it implies that a matrix A_{ij} derived from the integral equation of scattering can be factored as

$$A_{ij} = \mathbf{V}_{il}^t \cdot \overline{\boldsymbol{\alpha}}_{ll'} \cdot \mathbf{V}_{l'j}. \tag{8}$$

The above factorization is achievable by using the addition theorem, where l' corresponds to the center of the l'-th group, which contains the j-th element. This is possible for i and j belonging to different non-overlapping groups. Unfortunately, in the above, a scalar number A_{ij} is converted into a product of a vector, a matrix and a vector. Therefore, even though the number of "routes" diminishes as shown in Figure 8, the number of operations is not reduced — it is still of $O(N^2)$. It can be shown that the dimensions of $\mathbf{V}$ and $\overline{\boldsymbol{\alpha}}$ in (7) are proportional to M, the number of elements in the group it represents. Fortunately, a change of basis to the plane-wave basis diagonalizes the matrix $\overline{\boldsymbol{\alpha}}_{ll'}$. This diagonalization was first achieved by Rokhlin [44]. Hence, one can write

$$A_{ij} = \widetilde{\mathbf{V}}_{il}^t \cdot \widetilde{\overline{\boldsymbol{\alpha}}}_{ll'} \cdot \widetilde{\mathbf{V}}_{l'j} \tag{9}$$

where $\widetilde{\overline{\boldsymbol{\alpha}}}_{ll'}$ is now a diagonal matrix. By so doing, the number of operations for a matrix-vector multiply as expressed by (7) can be reduced for the non-nearest neighbor (non-overlapping) groups. Choosing the group size $M \sim \sqrt{N}$, the matrix-vector multiply can be effected in $O(N^{1.5})$ operations [44,45,85,86]. Figure 9 shows the use of the FMM to calculate the electromagnetic scattering of a NASA almond [86].

7. Ray-Propagation Fast Multipole Algorithm (RPFMA)

In the FMM, a matrix-vector multiply is expressed as

$$\sum_{j=1}^{N} A_{ij} x_j = \widetilde{\mathbf{V}}_{il}^t \cdot \sum_{l'=1}^{N/M} \widetilde{\overline{\boldsymbol{\alpha}}}_{ll'} \cdot \sum_{j \epsilon \mathcal{G}_{l'}} \widetilde{\mathbf{V}}_{l'j} x_j, \quad \begin{cases} i \in \mathcal{G}_l, \\ l = 1, \cdots, N/M. \end{cases} \tag{10}$$

The first step

$$\mathbf{b}_{l'} = \sum_{j \epsilon \mathcal{G}_{l'}} \widetilde{\mathbf{V}}_{l'j} x_j \tag{11}$$

calculates the plane waves with k-vectors on a sphere (or a circle in 2D) radiated by the sources x_j in group $\mathcal{G}_{l'}$. Then the second step

$$\mathbf{c}_l = \sum_{l'} \widetilde{\overline{\boldsymbol{\alpha}}}_{ll'} \cdot \mathbf{b}_{l'} \tag{12}$$

calculates the plane waves with different k-vectors on a sphere received by group $\mathcal{G}_l$ after the plane waves have been translated through the space separating the centers of groups $\mathcal{G}_l$ and $\mathcal{G}_{l'}$. Then the last multiply,

$$d_i = \widetilde{\mathbf{V}}_{il}^t \cdot \mathbf{c}_l \tag{13}$$

redistributes the plane waves received by group $\mathcal{G}_l$ to the i-th element of the group.

If the groups $\mathcal{G}_l$ and $\mathcal{G}_{l'}$ are far apart, it is clear that not all plane waves on a sphere will participate in the interaction between the elements of the two groups [87,88]. In fact, ray physics dictates that only a small fraction of the plane waves accounts for the interaction between the two groups. Therefore, the dimension of the matrix $\widetilde{\overline{\boldsymbol{\alpha}}}_{ll'}$ can be reduced to manifest this ray physics. Then the cost of the operation in (10) can be further reduced. Because of this, more intragroup calculation is desired. In this case, we choose $M \sim N^{1/3}$, and the complexity of a matrix-vector multiply can be further reduced to $O(N^{4/3})$.

A simplification of RPFMA is the fast far-field approximation (FAFFA) [89]. This method greatly simplifies the matrix elements for the far interactions between the elements; hence, they can be computed as needed. Therefore, an algorithm with $O(N)$ memory requirement is possible in this case.

8. Multilevel Algorithms

A logical extension of the two-level FMM is a multilevel algorithm [90-93]. In this case, the number of levels is proportional to $\log N$. If only N operations are needed at each level, this becomes an $N \log N$ algorithm for matrix-vector multiplies. Order N operations can be maintained at each level by interpolation and anterpolation [89,90]. Figure 10 shows the use of the multilevel fast multipole algorithm (MLFMA) to solve a 110,592-unknown problem on a workstation using the combined field integral equation [94]. The memory requirement of this algorithm is $O(N \log N)$, allowing large problems to be solved on a small computer.

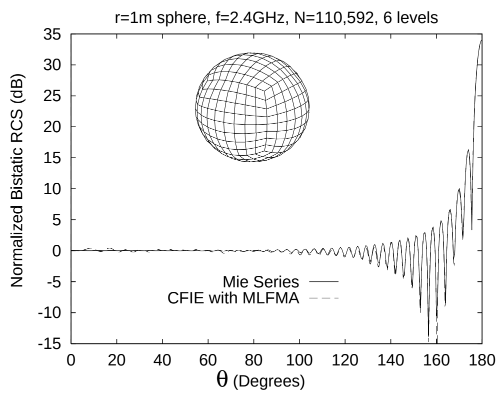

Figure 10. *Validations of CFIE [94] with MLFMA against the Mie series of the bistatic RCS of a metallic sphere of radius 1 m at 2.4 GHz (λ_0=12.5 cm) for VV polarization. 110,592 unknowns with a 6-level FMM are used. The RCS is normalized by πa^2. The computation takes 24 hours on a 10 MFLOPS machine.*

The matrix decomposition algorithm (MDA) and its multilevel cousin (MLMDA) [95,96] accelerate the iterative solution of electromagnetic scattering problems involving large scatterers. Unlike the FMM which relies on an analytical diagonalization of the translation operator, the MDA and MLMDA decompose MOM matrices using commonly available linear algebraic techniques. The MDA and MLMDA directly exploit the limited number of Degrees of Freedom (DOF) [97] that characterize a field observed over a domain that is "well-separated" from a source domain.

The MLMDA differs from the MDA in that off-diagonal blocks of the MOM matrix are aggregated into larger entities and decomposed using a multilevel algorithm that resembles an FFT as shown in Figure 11. The memory requirements and the computational complexity of the MDA are $O(N^{3/2})$ while those for the MLMDA asymptotically approach $N \log^2 N$. The MLMDA can easily be incorporated into existing Method of Moment (MOM) programs. The MLMDA has been applied to the solution of scattering problems involving large 2D scatterers with 50,000 unknowns.

9. Fast Steepest Descent Path Algorithm (FASDPA)

The Fast Steepest Descent Path Algorithm (FASDPA) [98,99] constitutes a novel two-level algorithm that hybridizes the FMM and the MDA. Not unlike the FMM and the MDA, the FASDPA starts from a spatial decomposition of the scatterer into a large number of subscatterers, and interactions between nearby subscatterers are accounted for directly. Interactions between distant subscatterers are expressed in terms of a small set of equivalent sources that exhaust the degrees of freedom of the interaction field, as in the MDA. To permit the algorithm to "recycle" information in a manner similar to the FMM, the field radiated by each group is represented in terms of a set of homogeneous plane waves. Equivalent source amplitudes

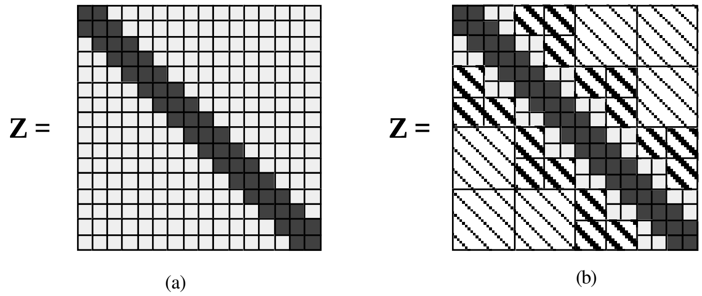

Figure 11. *The MOM matrix (a), is decomposed as shown in (b). Blocks representing near-field interactions are stored classically, while blocks representing far-field interactions are stored as products of low rank matrices (MDA) or are aggregated and stored using an FFT-like scheme.*

are obtained from the plane wave spectrum. More specifically, the FASDPA expresses the interaction field between distant groups as Equation (8). However, in contrast to the FMM, where the $\overline{\boldsymbol{\alpha}}_{ll'}$ matrix represents a diagonal translation matrix for homogeneous plane waves emanating from the source, the $\overline{\boldsymbol{\alpha}}_{ll'}$ matrix for the FASDPA is empty except for a small translation block, appearing on the diagonal. The computational complexity of the FASDPA is $O(N^{4/3})$ per iteration without proceeding to a multilevel scheme. Figure 12 compares the radar cross section of a corrugated semicircular structure computed using the FASDPA with results obtained using the MLMDA.

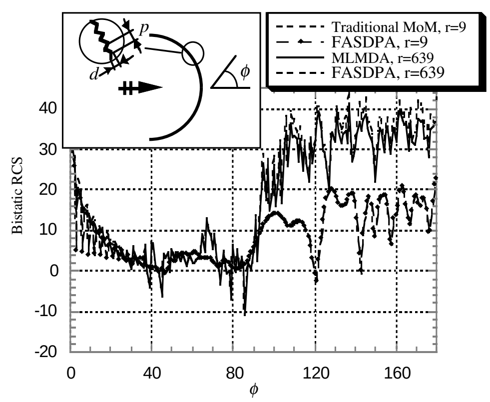

Figure 12. *Bistatic RCS of corrugated semicircular structure $p = 1.57\lambda$, their depth is $d = 0.5\lambda$, and the structure is illuminated by a TM_z plane wave traveling along the direction; $r = 9\lambda$ (N=400) and $r = 639\lambda$ (N=25,600).*

10. Fast Algorithm for Elongated Structures

Numerical algorithms for analyzing electromagnetic scattering from elongated objects, i.e., structures whose dimensions extend primarily along one spatial axis and which are uniform or of limited extent along the other two axes, are of great practical interest. A non-exclusive list of potential applications includes the analysis of scattering from rough surfaces, winglike structures, truncated and quasi-periodic structures, as well as the analysis of radiation from large phased-array antennas. Several methods have been proposed to seek an efficient solution to such problems [100,101].

The equivalent source algorithm for elongated structures (ESAES) [102] is a fast direct solver for analyzing scattering from such structures. ESAES is conceptually similar to RATMA for planar structures [100]: both are based on a recursive characterization of increasingly larger subscatterers using scattering matrices of reduced dimensions, and both algorithms have $N \log^2 N$ computational complexity and $O(N \log N)$ memory requirements. The ESAES abandons the plane-wave representation of the two-dimensional Green's function employed in [100] in favor of a reduced spatial representation of the fields that are scattered by an elongated object. This reduced spatial representation permits the computation of the fields radiated by N quasi-aligned sources and observed over an elongated domain, in terms of that radiated by $O(N \log N)$ equivalent sources. The concept of a reduced field representation is directly related to that of the limited number of degrees of freedom that characterize fields radiated by electromagnetic sources [97] . This reduced field representation can be obtained by augmenting an existing MOM code with purely algebraic techniques, e.g., a singular value or a rank revealing QR decomposition. We have applied the ESAES to 2D structures that measure several thousand wavelengths in length. Figure 13 shows the bistatic RCS of a finite periodic structure computed using the ESAES and compares the results to those obtained using the MLMDA.

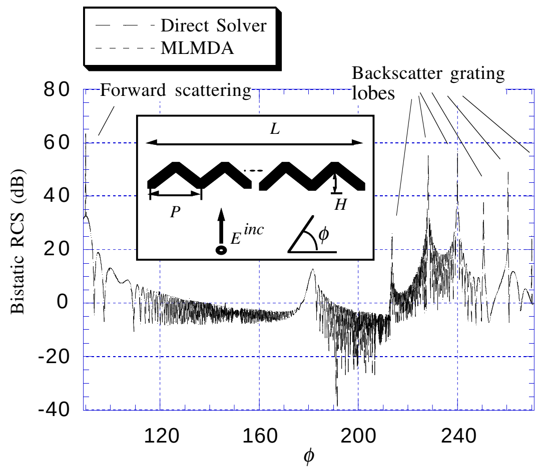

Figure 13. *Bistatic RCS of a finite triangular grating of length $L = 594\lambda$, $P = 6\lambda$, and $H = 1.1\lambda$ for normal TM plane wave incidence. Backscatter grating lobes can be observed.*

11. Adaptive Integral Method

Even though precorrected FFT methods have been used in the past to solve electrodynamic [71,76] and electrostatic problems [103], a note is in order on a related technique developed by Bleszynski et al [77], termed the adaptive integral method (AIM), which has been successfully applied to the analysis of scattering from very complex structures.

As in the FMM, the AIM separately considers near- and far-field interactions when evaluating a matrix-vector multiply. To compute far-field interactions, sources supported by the scatterer are projected onto a regular grid by matching their multipole moments (up to a certain order) in order to guarantee the approximate equality of their far-fields. Next, the fields at other grid locations produced by these grid-projected currents are evaluated by a three-dimensional convolution. Knowledge of these fields permits the computation of fields on the scatterer through interpolation. The projection and interpolation operators are represented by sparse matrices, while the convolution can be effected using an FFT. Unfortunately, the near-fields radiated by these grid-currents do not match those radiated by the original sources. Therefore, near-field interactions are evaluated directly, and corrected for errors introduced by the far-field operator.

For volumetric scatterers, the computational and memory costs associated with the AIM scale as $O(N \log N)$ and $O(N)$, respectively. For surface scatterers, its computational complexity scales as $O(N^{1.5} \log N)$ and its memory requirements as $N^{1.5}$. The computational complexity and memory requirements of the MLFMA are $O(N \log N)$, and hence asymptotically scale more favorably than those of the AIM. Nonetheless, the AIM competes with the MLFMA because the FFT butterfly tree is devoid of the complex interpolation and anterpolation operators inherent in MLFMA. Also, the AIM concept is applicable to all problems exhibiting convolutional structure, and is easier to implement than MLFMA. As a result, the AIM has been successfully applied to the analysis of scattering from very large 3D structures.

12. Conclusions

We have reviewed fast solution methods for efficiently solving electromagnetic scattering problems. Fast solution methods for electromagnetic scattering problems will have a definite impact in the area of computer-aided design of many technologies that rely on Maxwell's equations.

Even though a matrix-vector multiply for scattering problems only requires $O(N \log N)$ operations both for volume scattering and surface scattering problems, the number of iterations needed remains unpredictable. Therefore, preconditioning techniques for reducing the required number of iterations in iterative methods are urgently needed in solving electromagnetic wave scattering problems. Finally, even though direct solvers with reduced computational complexities are available for volumetric scattering problems, no such solvers exist for surface scatterers, except for collinear (or almost coplanar) structures. Hence, this remains an open problem.

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