Complex Coordinate System as a Generalized Absorbing Boundary Condition †

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Abstract

By a change of variables, we show that Maxwell's equations for PML media reduce to ordinary Maxwell's equations but with complex coordinate systems. Many closed form solutions for Maxwell's equations map to corresponding closed form solutions in complex coordinate systems. Numerical simulations with the closed form solutions show that metallic boxes lined with PML media are highly absorptive lending a better insight into the absorptive properties of PML media. More importantly, the complex coordinate system method can be easily generalized to non-Cartesian coordinate systems, providing absorbing boundary conditions in these coordinate systems.

1. Introduction

The perfectly matched layer (PML) as a material absorbing boundary condition was first proposed by Berenger [1]. Later, Katz et al [2] extended it to three dimensions, while Chew and Weedon [3] proposed a coordinate stretching viewpoint. An anisotropic media interpretation has also been proposed by Zack et al [4]. Due to its efficacy as a material absorbing boundary condition, it has been fervently studied by many workers for numerical simulations of PDE's [5-13].

In this paper, we expand on the coordinate stretching viewpoint, and introduce a change of variables to transform Maxwell's equations in PML media into ordinary-looking Maxwell's equations in a complex coordinate system. As a result, many closed form solutions that have already existed can be easily mapped into closed form solutions in these complex coordinate systems. Moreover, the method can be easily generalize to non-Cartesian coordinate systems.

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2. Complex Coordinate System

Maxwell's equations in a stretched coordinate system are given by [3]

$$\nabla_\sigma \times \mathbf{E} = i\omega\mu\mathbf{H}, \tag{1}$$

$$\nabla_\sigma \times \mathbf{H} = -i\omega\epsilon\mathbf{E}, \tag{2}$$

$$\nabla_\sigma \cdot \epsilon\mathbf{E} = 0, \tag{3}$$

$$\nabla_\sigma \cdot \mu\mathbf{H} = 0, \tag{4}$$

where

$$\nabla_\sigma = \hat{x}\frac{1}{s_x}\frac{\partial}{\partial x} + \hat{y}\frac{1}{s_y}\frac{\partial}{\partial y} + \hat{z}\frac{1}{s_z}\frac{\partial}{\partial z}. \tag{5}$$

In the above, $s_i, i = x, y, z$ are coordinate stretching variables. When s_i's, where $i = x, y, z$, are functions of i only, then (3) and (4) are derivable from (1) and (2).

With the following change of variables,

$$\tilde{x} = \int_0^x s_x(x')\,dx', \tag{6a}$$

$$\tilde{y} = \int_0^y s_y(y')\,dy', \tag{6b}$$

$$\tilde{z} = \int_0^z s_z(z')\,dz', \tag{6c}$$

the ∇_σ operator becomes

$$\tilde{\nabla} = \hat{x}\frac{\partial}{\partial\tilde{x}} + \hat{y}\frac{\partial}{\partial\tilde{y}} + \hat{z}\frac{\partial}{\partial\tilde{z}}, \tag{7}$$

because

$$\frac{\partial}{\partial\tilde{x}} = \frac{1}{s_x}\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial\tilde{y}} = \frac{1}{s_y}\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial\tilde{z}} = \frac{1}{s_z}\frac{\partial}{\partial z}. \tag{8}$$

In this case, Equations (1) to (4) just become ordinary looking Maxwell's equations, except that $\tilde{x}$, $\tilde{y}$, and $\tilde{z}$ can be complex valued as well. As a result, many closed form solutions can be obtained for PML media using the mapping provided by (6a) to (6c).

When $\tilde{x}$, $\tilde{y}$, and $\tilde{z}$ are complex, the wave that travels through the complex coordinates is attenuated, but not reflected. More importantly, the complex coordinate system for Maxwell's equations can be written in any coordinate system with complex variables as the coordinate variables. We will illustrate the solutions in the complex Cartesian, cylindrical, and spherical coordinates.

3. Cartesian Coordinates

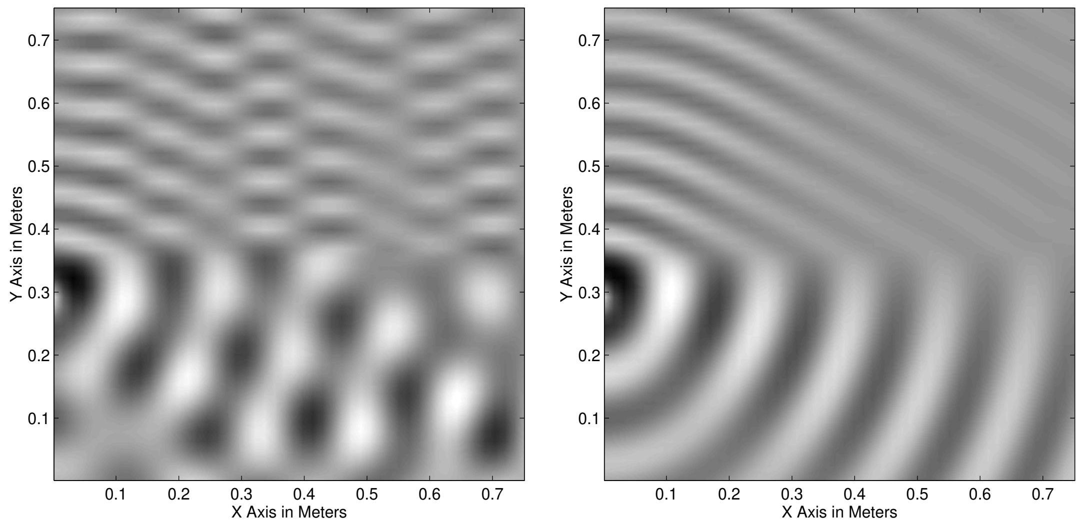
We will solve for the closed form solution of a line source inside a metallic box of width a, and truncated in the vertical direction at $y = b$. The box is filled with two half spaces with different wave numbers k_1 and k_2, and the source is located at (x', y'), with a dielectric interface at $y = h$. The box is assumed to be infinitely extended in the negative y direction. The solution is given by [for detail of the solution technique, see 14]

$$\psi(x, y) = -\sum_{n=0}^{\infty} X_n(x)X_n(x')Y_n(y, y') \tag{9a}$$

for $-\frac{a}{2} < x < \frac{a}{2}$, $y < b$, where

$$X_n(x) = \sqrt{\frac{2}{a}}\sin\left[k_{nx}\left(x + \frac{a}{2}\right)\right], \quad -\frac{a}{2} < x < \frac{a}{2}, \tag{9b}$$

$$Y_n(y, y') = \begin{cases} \frac{1}{2i\gamma_1}\left[e^{i\gamma_1|y-y'|} + \tilde{R}_{12}e^{-i\gamma_1(y-h)+i\gamma_1|h-y'|}\right], & y < h, \\ \frac{1}{2i\gamma_1}\tilde{T}_{12}\left[e^{i\gamma_2(y-h)} - e^{-i\gamma_2(y-2b+h)}\right], & b > y > h, \end{cases} \tag{9c}$$



(a) **(b)**

Figure 1. The solution of a line source in a two-dimensional box with $a = b = 1.5$ *m. The frequency is* 2.2 *GHz,* $\epsilon_1 = 1$, $\epsilon_2 = 4$, $\Delta = 0.15$ *m,* $h = 0.25b$, *and* $(x', y') = (0, 0.2b)$. *(a)* $\alpha = 0$ *and one can see many reflections from the walls of the box. (b)* $\alpha = 2$, *and little reflection is observed.*

$$k_{nx} = \frac{n\pi}{a}, \quad \gamma_1 = \sqrt{k_1^2 - k_{nx}^2}, \quad \gamma_2 = \sqrt{k_2^2 - k_{nx}^2}. \tag{9d}$$

$\tilde{R}_{12}$ and $\tilde{T}_{12}$ are generalized reflection and transmission coefficients, respectively, at the upper and lower interfaces [14]. To derive the solution in the complex coordinate system, we make use of Equations (6), and define

$$s_x(x) = \begin{cases} 1, & 0 < x < x_m, \\ 1 + i\alpha\left(\frac{x - x_m}{0.5a - x_m}\right)^2, & x_m < x < \frac{a}{2}, \end{cases} \tag{10}$$

where $x_m = 0.5a - \Delta$ and Δ is a small real positive number. In this manner, in accordance to (34), we obtain the mapping

$$\tilde{x} = \begin{cases} x, & 0 < x < x_m, \\ x + i\alpha\frac{(x - x_m)^3}{3(0.5a - x_m)^2}, & x_m < x < \frac{a}{2}. \end{cases} \tag{11}$$

In other words, $\tilde{x}$ assumes complex values in this complex coordinate stretching or map. The mapping is linear for most part except when $x \approx a/2$. (This map is chosen as an illustration, because it resembles the parabolic profile often chosen in PML simulations. Other maps are also possible.) A similar map occurs for $x < 0$. Hence, in the complex coordinate system, the boundary of the box is defined by $\tilde{a}$ which is complex valued. Similar mapping can be defined for the y-axis such that $y \mapsto \tilde{y}$, $b \mapsto \tilde{b}$, where $\tilde{y}$ and $\tilde{b}$ are complex valued.

Figure 1 shows the above solution (a) with, (b) without complex coordinate mapping. In (a) ($\alpha = 0$, one clearly sees the reflection from the walls. (Only the domain $0 < x < 0.5a$ and $0 < y < 0.5b$ is plotted.) In case (b) ($\alpha = 2$), the reflection from the wall is greatly diminished. In this case, as shown by Equation (11), the boundary of the box is located in a complex space. This result explains why PML remains highly absorptive (a) when a dielectric interface extends into the PML media, and (b) at the corner region of a simulation box laced with PML.

4. Cylindrical Coordinates

The solution of a line source in a cylinder with a perfectly conducting wall is [14]

$$\psi(\rho,\phi) = \frac{i}{4} H_0^{(1)}(k|\boldsymbol{\rho}-\boldsymbol{\rho}'|) - \frac{i}{4}\sum_{n=-\infty}^{\infty} J_n(k\rho)\frac{J_n(k\rho')H_n^{(1)}(ka)}{J_n(ka)} e^{in(\phi-\phi')}. \tag{12}$$

We can define a map

$$\tilde{\rho} = \begin{cases} \rho, & 0<\rho<\rho_m, \\ \rho + i\alpha\frac{(\rho-\rho_m)^3}{3(a-\rho_m)^2}, & \rho_m<\rho<a, \end{cases} \tag{13}$$

where $\rho_m = a - \Delta$. Figure 2 shows the calculation of the above series summation formula for different values of α. When $\alpha > 0$ (case (b)), complex coordinate mapping occurs near the wall according to Equation (13) and reflection is diminished.

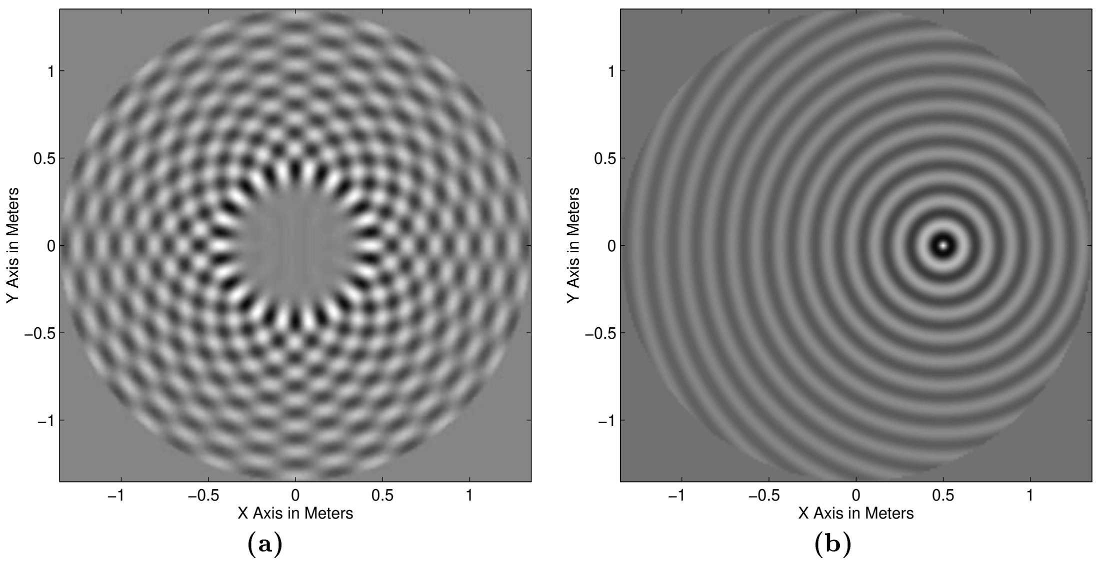


Figure 2. The solution of a line source in a circular cylinder of radius 1.5 *m where* $(x', y') = (0.5, 0.0)$ *m. The frequency is* 2.2 *GHz, and* $\Delta = 0.15$ *m. (a)* $\alpha = 0$*, and one can see many reflections from the wall of the cylinder. (b)* $\alpha = 2$*, and one sees little reflections.*

5. Spherical Coordinates

The solution of a point source in a sphere with a homogeneous Direchlet boundary condition is [14]

$$\psi(\mathbf{r}) = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} - ik\sum_{n=0}^{\infty}\sum_{m=-n}^{n} j_n(kr)\, Y_{nm}(\theta,\phi)\, Y^*_{nm}(\theta',\phi')\, \frac{j_n(kr')h_n^{(1)}(ka)}{j_n(ka)}, \tag{14}$$

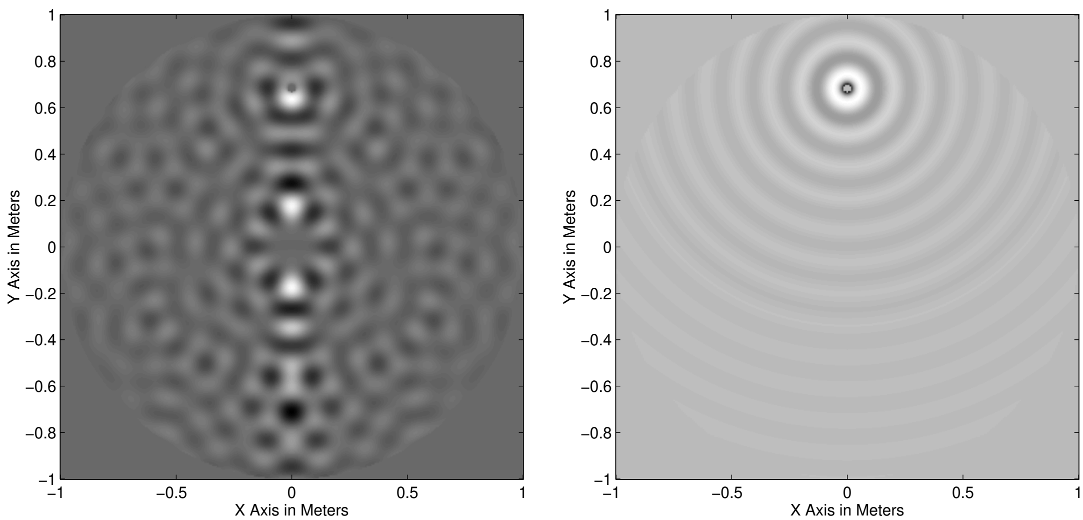
where

$$Y_{nm}(\theta,\phi) = \sqrt{\frac{(n-m)!}{(n+m)!}\frac{2n+1}{4\pi}}\; P_n^m(\cos\theta) e^{im\phi}. \tag{15}$$

The corresponding complex map is

$$\tilde{r} = \begin{cases} r, & 0 < r < r_m, \\ r + i\alpha \frac{(r-r_m)^3}{3(a-r_m)^2}, & r_m < r < a, \end{cases} \tag{16}$$

where $r_m = a - \Delta$. Figure 3 shows the calculation of the above series summation formula. When $\alpha > 0$ (case (b)), complex coordinate stretching occurs near the wall according to Equation (16) and little reflection is observed.



(a) **(b)**

Figure 3. The solution of a point source in a sphere of radius 1.0 *m with* $(x', y') = (0.0, 0.75)$ *m. The frequency is* 2.2 *GHz, and* $\Delta = 0.15$ *m. (a)* $\alpha = 0$*, and one can see much reflections from the wall of the sphere. (b)* $\alpha = 2$*, and little reflection is observed.*

6. Conclusions

We have demonstrated that PML in Cartesian coordinates is equivalent to solving the wave equation or Maxwell's equations in a complex coordinate system. Therefore, closed form solutions that exist in the real coordinate system map to solutions in the complex coordinate system. In the complex coordinate system, the boundaries exist in a complex space, providing absorbing boundary conditions. Hence, this transformation provides a new view of PML in the Cartesian coordinates, clearly showing that a mapping to a complex coordinate system does not induce reflections, explaining why PML works near the corner of a simulation region, and when a dielectric interface, or a metallic surface, extends to the edge of a simulation region.

The idea of a complex coordinate system can be further generalized to other curvilinear coordinate system. We have demonstrated the solutions and their absorbing boundary conditions in cylindrical as well as spherical coordinates.

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