FAST MULTIPOLE METHOD SOLUTION OF COMBINED FIELD INTEGRAL EQUATION †

J. M. Song and W. C. Chew
Electromagnetics Laboratory
Department of Electrical and Computer Engineering
University of Illinois
Urbana, IL 61801

ABSTRACT—- The electromagnetic (EM) scattering from three-dimensional (3D) conducting bodies has been studied. The electric field integral equation (EFIE), magnetic field integral equation (MFIE), and combined field integral equation (CFIE) are approximated by matrix equations using the method of moments (MOM). The fast multipole method (FMM) has been implemented to speed up the matrix-vector multiply in the CG method when it is used to solve EFIE, MFIE, and CFIE. FMM reduces the complexity of a matrix-vector multiply from $O(N^2)$ to $O(N^{1.5})$. With a multilevel fast multipole algorithm (MLFMA), the complexity is further reduced to $O(N\log N)$.

1. Introduction

The EM field scattering by arbitrarily shaped conductor can be obtained by finding the solution of an integral equation where the unknown function is the induced current distribution. The integral equation is discretized into a matrix equation by the method of moments (MOM). The resultant matrix equation is then solved by Gaussian elimination, which requires $O(N^3)$ floating-point operations if Gaussian elimination is used to solve N linear equations, or $O(N^2)$ operations per iteration if the conjugate gradient (CG) method is used.

The fast multipole method (FMM) [1-6] speeds up the matrix-vector multiply in the conjugate gradient (CG) method when it is used to solve the matrix equation iteratively. In this paper, FMM is applied to solve the electromagnetic scattering from three dimensional arbitrary shape conducting bodies. The electric field integral equation (EFIE), magnetic field integral equation (MFIE), and combined field integral equation (CFIE) are considered. FMM formula for CFIE has been derived, which reduces the complexity of a matrix-vector multiply from $O(N^2)$ to $O(N^{1.5})$, where N is the number of unknowns. With a nonnested method, using the ray-propagation fast multipole algorithm (RPFMA) [5,6], the cost of a FMM matrix-vector multiply is reduced to $O(N^{4/3})$. We have implemented a multilevel fast multipole algorithm (MLFMA), whose complexity is further reduced to $O(N\log N)$. The FMM also requires less memory, and hence, can solve a larger problem on a small computer.

2. The Combined Field Integral Equation (CFIE)

Practical electromagnetic problems are often three-dimensional and involve arbitrary geometry. The arbitrary surface is described by dividing it into a number of connected patches which are mathematically described as parametric quadratic surfaces [6,7]. For conducting objects, the electric field integral equation (EFIE) is given by

---

† This work was supported by NASA under grant NASA NAG 2-871, Office of Naval Research under grant N00014-89-J1286, the Army Research Office under contract DAAL03-91-G-0339, and the National Science Foundation under grant NSF ECS 92-24466. To appear in the ACES Conference Digest, March 1995.

$$\hat{t} \cdot \int_S \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}(\mathbf{r}') dS' = \frac{4\pi i}{k\eta} \hat{t} \cdot \mathbf{E}^i(\mathbf{r}), \tag{1}$$

for $\mathbf{r}$ on surface S, where $\hat{t}$ is any unit tangent vector on S, and

$$\overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') = (\bar{\mathbf{I}} - \frac{1}{k^2} \nabla \nabla') g(\mathbf{r}, \mathbf{r}'), \tag{2}$$

$$g(\mathbf{r}, \mathbf{r}') = \frac{e^{ikR}}{R}, \quad R = |\mathbf{r} - \mathbf{r}'|. \tag{3}$$

For closed conducting objects, the magnetic field integral equation (MFIE) is given by

$$2\pi \mathbf{J}(\mathbf{r}) - \hat{n} \times \nabla \times \int_S dS' g(\mathbf{r}, \mathbf{r}') \mathbf{J}(\mathbf{r}') = 4\pi \hat{n} \times \mathbf{H}^i(\mathbf{r}) \tag{4}$$

for $\mathbf{r}$ approaches to S from outside, where $\hat{n}$ is an outwardly directed normal. MFIE (4) can be written as

$$2\pi \hat{t} \cdot \mathbf{J}(\mathbf{r}) - \hat{t} \cdot \hat{n} \times \nabla \times \int_s dS' g(\mathbf{r}, \mathbf{r}') \mathbf{J}(\mathbf{r}') = 4\pi \hat{t} \cdot \hat{n} \times \mathbf{H}^i(\mathbf{r}). \tag{5}$$

Then combined field integral equation (CFIE) for closed conducting objects is simply a linear combination of EFIE (1) and MFIE (5) and is of the form:

$$\alpha \text{EFIE} + (1 - \alpha) \frac{i}{k} \text{MFIE}. \tag{6}$$

The combination parameter α ranges from 0 to 1 and can be chosen to be any value within this range. It is found $\alpha = 0.2$ to be an overall good choice.

Let us apply the method of moments (MOM) to the CFIE. The unknown current $\mathbf{J}(\mathbf{r})$ is first expanded in an appropriately chosen set of basis functions $\{\mathbf{j}_i\}$. The CFIE for $\mathbf{J}(\mathbf{r})$ is discretized by substituting the expansion in terms of unknowns a_n. Then, rather than forcing the CFIE to be satisfied for $\mathbf{r}$ approaching S from the outside, it is multiplied by a set of N testing functions $\{\mathbf{t}_j\}$ and the inner products are taken:

$$\sum_{i=1}^{N} A_{ji} a_i = F_j, \qquad j = 1, 2, \cdots, N, \tag{7}$$

where

$$\begin{aligned} A_{ji} &= \alpha \int_S dS \mathbf{t}_j(\mathbf{r}) \cdot \int_S dS' \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{j}_i(\mathbf{r}') \\ &+ (1 - \alpha) \left[\frac{2\pi i}{k} \int_S dS \mathbf{t}_j(\mathbf{r}) \cdot \mathbf{j}_i(\mathbf{r}) - \frac{i}{k} \int_S dS \mathbf{t}_j(\mathbf{r}) \cdot \hat{n} \times \nabla \times \int_S dS' g(\mathbf{r}, \mathbf{r}') \mathbf{j}_i(\mathbf{r}') \right]. \end{aligned} \tag{8}$$

Finally, the integral equations are approximated by matrix equations (7) using the method of moments (MOM). The specially designed basis functions for subdomains which contain surface curvature are the generalized "rooftop" basis functions with a constant charge density and without line charge accumulation on the surfaces [7].

3. The Fast Multipole Method (FMM)

Letting $\mathbf{r}_j$ and $\mathbf{r}_i$ be the field point and source point, respectively, we have

$$\mathbf{r}_{ji} = \mathbf{r}_j - \mathbf{r}_i = \mathbf{r}_j - \mathbf{r}_m + \mathbf{r}_m - \mathbf{r}_{m'} + \mathbf{r}_{m'} - \mathbf{r}_i = \mathbf{r}_{jm} + \mathbf{r}_{mm'} - \mathbf{r}_{im'}. \tag{9}$$

Thus, using the addition theorem [2,8] the scalar Green's function can be rewritten as

$$\frac{e^{ikr_{ji}}}{r_{ji}} = \frac{ik}{4\pi} \int d^2\hat{k} e^{i\mathbf{k}\cdot(\mathbf{r}_{jm}-\mathbf{r}_{im'})} \alpha_{mm'}(\hat{r}_{mm'} \cdot \hat{k}), \tag{10}$$

where

$$\alpha_{mm'}(\hat{r}_{mm'} \cdot \hat{k}) = \sum_{l=0}^{L} i^l (2l+1) h_l^{(1)}(kr_{mm'}) P_l(\hat{r}_{mm'} \cdot \hat{k}). \tag{11}$$

We have truncated the infinite summation in (11). The integration $\int d^2\hat{k}$ in (10) represents the integrals over the unit sphere, and will be evaluated by Gaussian quadratures with $K = 2L^2$ points. Substituting the scalar Green's function (10) into the dyadic function (2) leads to

$$\overline{\mathbf{G}}(\mathbf{r}_j, \mathbf{r}_i) = \frac{ik}{4\pi} \int d^2\hat{k} (\overline{\mathbf{I}} - \hat{k}\hat{k}) e^{i\mathbf{k}\cdot(\mathbf{r}_{jm}-\mathbf{r}_{im'})} \alpha_{mm'}(\hat{r}_{mm'} \cdot \hat{k}). \tag{12}$$

Substituting Equations (10) and (12) into (8), we get a formula to calculate the matrix elements in non-nearby group pairs,

$$A_{ji} = \frac{ik}{4\pi} \int d^2\hat{k} \mathbf{V}_{fmj}(\hat{k}) \cdot \alpha_{mm'}(\hat{k} \cdot \hat{r}_{mm'}) \mathbf{V}^*_{sm'i}(\hat{k}), \tag{13}$$

where

$$\mathbf{V}_{fmj}(\hat{k}) = \alpha \int_S dS e^{i\mathbf{k}\cdot\mathbf{r}_{jm}} (\overline{\mathbf{I}} - \hat{k}\hat{k}) \cdot \mathbf{t}_j(\mathbf{r}_{jm}) - (1-\alpha)\hat{k} \times \int_S dS e^{i\mathbf{k}\cdot\mathbf{r}_{jm}} \mathbf{t}_j(\mathbf{r}_{jm}) \times \hat{n}, \tag{14}$$

$$\mathbf{V}_{sm'i}(\hat{k}) = \int_S dS e^{i\mathbf{k}\cdot\mathbf{r}_{im'}} (\overline{\mathbf{I}} - \hat{k}\hat{k}) \cdot \mathbf{j}_i(\mathbf{r}_{im'}). \tag{15}$$

When we use CG method to solve the matrix equations (7), matrix-vector multiply is needed to be computed in each iteration. Then, we can write the matrix-vector multiplication as

$$\sum_{i=1}^{N} A_{ji} a_i = \sum_{m' \in B_m} \sum_{i \in G_{m'}} A_{ji} a_i + \frac{ik}{4\pi} \int d^2\hat{k} \mathbf{V}_{fmj}(\hat{k}) \cdot \sum_{m' \notin B_m} \alpha_{mm'}(\hat{k} \cdot \hat{r}_{mm}) \sum_{i \in G_{m'}} \mathbf{V}^*_{sm'i}(\hat{k}) a_i. \tag{16}$$

The first term in (16) is the contribution from nearby groups (including the self-group), and the second term is the far interaction calculated by FMM. The FMM idea is first to divide the subscatterers into groups. Then, addition theorem is used to translate the scattered field of different scattering centers within a group into a single center. Hence, the number of scattering centers is reduced. Similarly, for each group, the field scattered by all the other group centers can be first "received" by the group center, and then "redistributed" to the subscatterers belonging to the group. It is proven that the computation cost using (16) with 2-level FMM is in order $O(N^{1.5})$ [6].

To implement a multilevel fast multipole algorithm (MLFMA), the entire object is first enclosed into a large cube, which is partitioned into eight smaller cubes. Each subcube is then recursively subdivided into smaller cubes until the edge length of the finest cube is about half of a wavelength. Cubes at all levels are indexed. At the finest level, we find which cube each basis function reside in by comparing the coordinates of the center of the basis function with the center of cube . We further find nonempty cubes by sorting. Only nonempty cubes are recorded using tree-structured data at all levels [9]. Thus, the computational cost depends on only the nonempty cubes, not all cubes.

The basic algorithm for matrix-vector multiply is broken down into two sweeps [10]: the first sweep consists of constructing outer multipole expansions for each nonempty cube at all levels. The second consists of constructing local multipole expansions contributed from the well-separated cubes at all levels. When the cube becomes larger from the finest level to the coarsest level, the numbers of multipole expansions should increase. In the first sweep, the outer multipole expansions are computed at the finest level, then the expansions for larger cube are obtained using shifting and interpolation. Let $\mathbf{r}_{m'_l}$ and $\mathbf{r}_{m'_{l-1}}$ be the cube centers at level l and $l-1$, respectively, then the outer multipole expansions for coarser level should be

$$\mathbf{V}_{sm'_{l-1}i}(\hat{k}) = e^{i\mathbf{k}\cdot\mathbf{r}_{m'_l m'_{l-1}}}\mathbf{V}_{sm'_l i}(\hat{k}). \tag{17}$$

But $\mathbf{V}_{sm'_l i}(\hat{k})$ has only K_l values, we need K_{l-1} values of $\mathbf{V}_{sm'_{l-1}i}(\hat{k})$. Then we will interpolate $\mathbf{V}_{sm'_l i}(\hat{k})$ to K_{l-1} values first, then

$$\mathbf{V}_{sm'_{l-1}i}(\hat{k}_{(l-1)n'}) = e^{i\mathbf{k}_{(l-1)n'}\cdot\mathbf{r}_{m'_l m'_{l-1}}}\sum_{n=1}^{K_l} W_{n'n}\mathbf{V}_{sm'_l i}(\hat{k}_{ln}), \tag{18}$$

where interpolation matrix W is a sparse matrix.

At the coarsest level, the local multipole expansions contributed from well-separated cubes are calculated using the second part of (16). At the second sweep, the local expansions for smaller cubes include the contributions from parent cube using shifting and anterpolation [11], and from the well-separated cube at this level but not well-separated at the parent level. If the local multipole expansions received by the cube center al level $l-1$ is $\mathbf{B}(\hat{k})$, then the contribution from all well-separated cubes can be written as

$$I = \int d^2\hat{k}\mathbf{V}_{fm_{l-1}j}(\hat{k})\cdot\mathbf{B}(\hat{k}) = \sum_{n'=1}^{K_{l-1}} w_{n'}\mathbf{V}_{fm_{l-1}j}(\hat{k}_{(l-1)n'})\cdot\mathbf{B}(\hat{k}_{(l-1)n'}), \tag{19}$$

where $w_{n'}$ is the weighting function. Substituting the interpolation expression for $\mathbf{V}_{fm_{l-1}j}(\hat{k}_{(l-1)n'})$ into (19), and changing the order of two summations lead to

$$I = \sum_{n=1}^{K_l} w_n\mathbf{V}_{fm_l j}(\hat{k}_{ln})\cdot\sum_{n'=1}^{K_{l-1}} W_{n'n}\mathbf{B}(\hat{k}_{(l-1)n'})e^{i\mathbf{k}_{(l-1)n'}\cdot\mathbf{r}_{m'_l m'_{l-1}}}w_{n'}/w_n. \tag{20}$$

The above operation is called anterpolation [11]. At the finest level, the contributions from non-well-separated cube are calculated directly. Since only nonempty cubes are considered, the complexity for MLFMA is further reduced to $O(N\log N)$.

4. Results and Conclusions

In Figure 1, we plot the normalized residual norm in CG method as functions of the number of iterations for EFIE, MFIE, and CFIE. It is found that CFIE converges much faster than EFIE and MFIE. EFIE and MFIE may not converge or converge too slowly at specific frequencies corresponding to the eigenvalues of the interior resonator problem, and give a wrong solution. This phenomenon is called the internal resonance corruption, because the tangential components of a single incident field (either the electric or magnetic field) at the scatterer surface lack sufficient information to uniquely determine the desired surface currents at a resonance frequency. It has been proven that the CFIE eliminates the effect of these resonances. At all frequencies, CFIE has an unique solution, and converges faster than EFIE and MFIE if the matrices are solved by an iteration method since the matrix for CFIE has smaller condition number than those for EFIE and MFIE.

Figure 2 shows the comparison of the CPU time versus the number of unknowns for the solution of CFIE with the FMM, standard CG, and LU decomposition for calculating the bistatic RCS of a conducting sphere. It is seen that the FMM outperforms the standard CG in terms of matrix filling and matrix solving. The FMM also requires less memory, and hence, can solve a larger problem on a small computer. The simulation is performed on a SUN-SPARC-2 with 64 MB of RAM. The final residual norm in CG method is less than 10^{-4}.

Figure 3 shows the validation of the numerical result from combined field integral equation (CFIE) with FMM against the Mie series solution of the bistatic RCS of a metallic sphere of radius 1m at frequency of 0.72GHz for the parallel polarization. 9,408 unknowns with 2-level FMM are used. The solutions of CFIE with FMM agree with Mie series very well.

Figure 4 shows the RCS of a one-meter long NASA almond at 2.5 GHz in the xy plane with $\theta = 90^o$. Five unknowns are used per wavelength. The calculation is done by using CFIE with LU decomposition on a SUN-SPARC-10 with 128 MB RAM, and it consumes about 24 hr of CPU time. Some points computed using CFIE with 2-level FMM are shown. These points took 0.5 hr/point to compute, and much less than 3 hr/point to compute with EFIE using FMM. The numerical results are compared with experimental measurement from Ohio State University [12]. There is a good agreement between our results and the measurement.

Figure 5 shows the bistatic RCS of a one-meter long metallic square plate at 4.5GHz in the xy plane with incident angle $\theta = 45^o$. 32,512 unknowns with 6-level FMM are used. It is done by solving EFIE on a SUN-SPARC-2 with 64MB RAM. The agreement between our results and the approximation by physical optics is good when the RCS is bigger than 0 dB.

In conclusion, the fast multipole method (FMM) has been implemented to speed up the matrix-vector multiply when CG method is used to solve EFIE, MFIE, and CFIE. At all frequencies, CFIE has an unique solution, and converges faster than EFIE and MFIE. FMM approach reduces the complexity of a matrix-vector multiply from $O(N^2)$ to $O(N^{1.5})$. With a multilevel fast multipole algorithm (MLFMA), the complexity is reduced to $O(N\log N)$. The FMM also requires less memory, and hence, can solve a larger problem on a small computer.

REFERENCES

[1] V. Rokhlin, "Rapid Solution of Integral Equations of Scattering Theory in Two Dimensions," *J. Comput. Phys*, vol. 86, no. 2, pp. 414-439, February 1990.

[2] R. Coifman, V. Rokhlin, and S. Wandzura, "The fast Multipole Method for the Wave Equation: A Pedestrian Prescription," *IEEE Antennas Propagat. Mag.*, vol. 35, no. 3, pp. 7-12, June 1993.

[3] C.C. Lu and W.C. Chew, "A Fast Algorithm for Solving Hybrid Integral Equation," *IEE Proceedings-H*, vol.140, no.6, pp.455-460, December 1993.

[4] B. Dembart and E. Yip, "A 3D Moment Method Code Based on Fast Multipole," *Digest of the 1994 URSI Radio Science Meeting*, p. 23, Seattle, Washington, June 1994.

[5] R.L. Wagner and W.C. Chew, "A Ray-Propagation Fast Multipole Algorithm," *Micro. Opt. Tech. Lett.*, vol.7, no.10, pp.435-438, July 1994.

[6] J.M. Song and W.C. Chew, "Fast Multipole Method Solution Using Parametric Geometry," *Micro. Opt. Tech. Lett.*, vol.7, no. 16, pp.760-765, November 1994.

[7] J.M. Song and W.C. Chew, "Moment Method Solution Using Parametric Geometry," *J. of Electromagnetic Waves and Appl.,* to be published.

[8] M. Abromowitz and I.A. Stegun, *Handbook of Mathematical Functions,* Dover Publication, New York, 1972.

[9] J. Barnes and P. Hut, "A Hierarchical $O(NlogN)$ Force-calculated Algorithm," *Nature*, Vol. 324, pp. 446-449, December 4, 1986.

[10] C. R. Anderson, "An Implementation of the Fast Multipole Method without Multipole," *SIAM J. Sci. Stat. Comput.*, Vol. 13, No. 4, pp. 923-947, July 1992.

[11] A. Brandt, "Multilevel Computations of Integral Transforms and Particle Interactions with Oscillatory Kernels," *Comput. Phys. Commun.,* Vol. 65, pp. 24-38, 1991.

[12] M.I. Sancer, R.L. McClary, and K.J. Glover, "Electromagnetic Computation Using Parametric Geometry," *Electromagnetics*, vol. 10, no. 1-2, pp. 85-103, 1990.

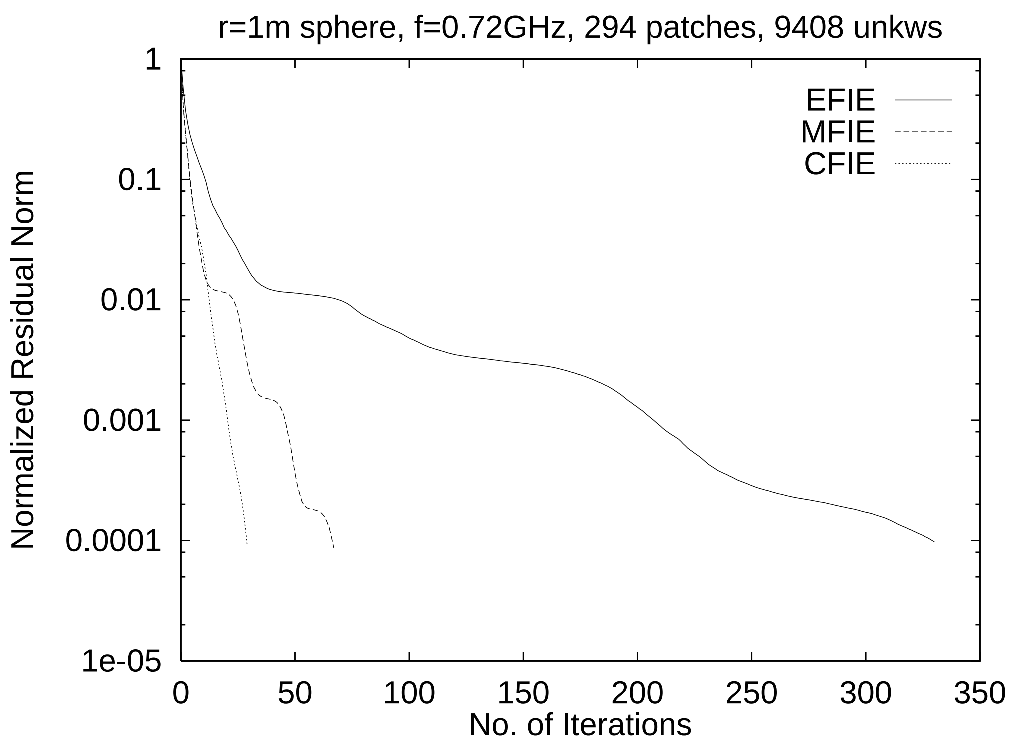

Figure 1. Comparison of the convergence of solutions of EFIE, MFIE, and CFIE for a conducting sphere of radius 1 m at 0.72 GHz for parallel polarization (9,048 unknowns, 7.2 unknowns per wavelength).

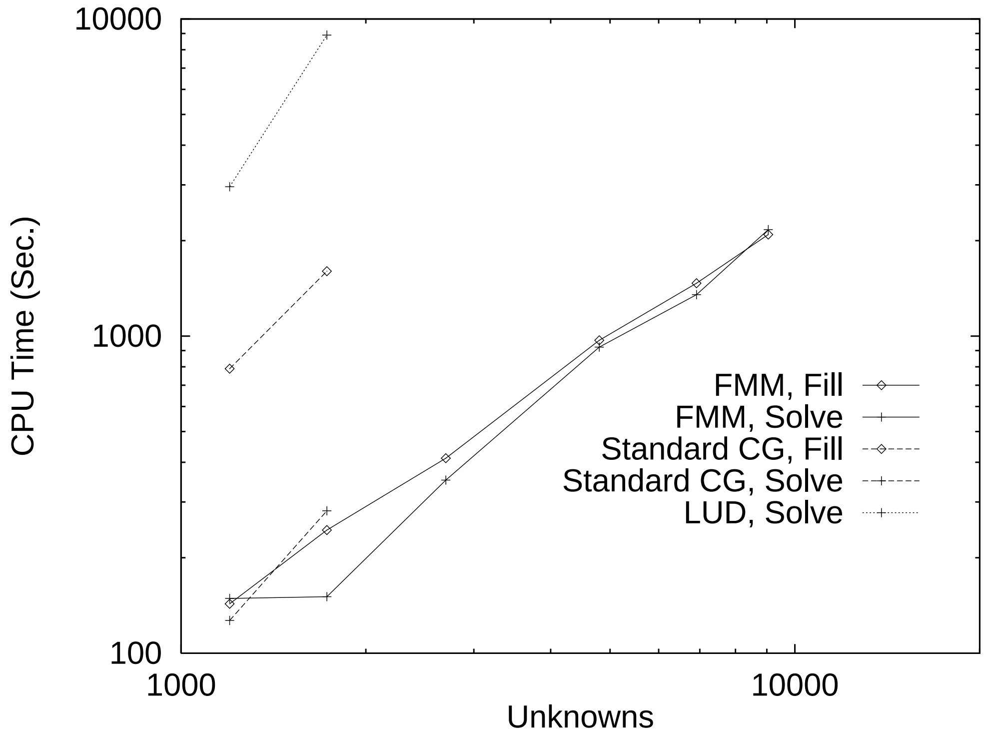

Figure 2. Comparison of the CPU time versus the number of unknowns for FMM, standard CG and LU decomposition for a metallic sphere.

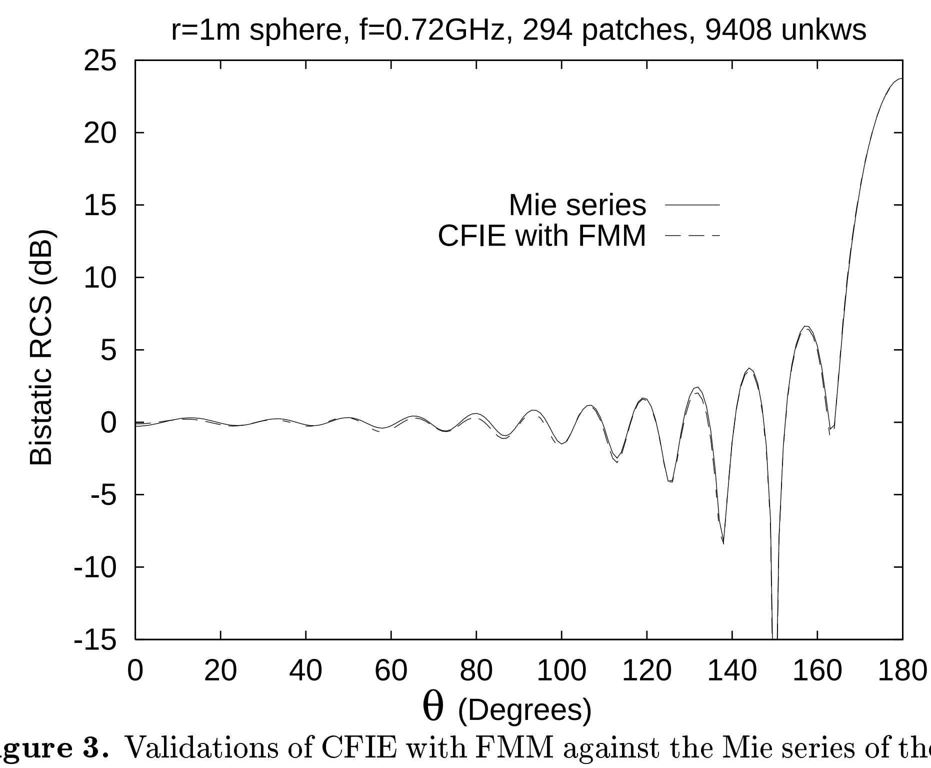

Figure 3. Validations of CFIE with FMM against the Mie series of the bistatic RCS of a metallic sphere of radius 1m at 0.72GHz for VV polarization. The RCS is normalized by πa^2.

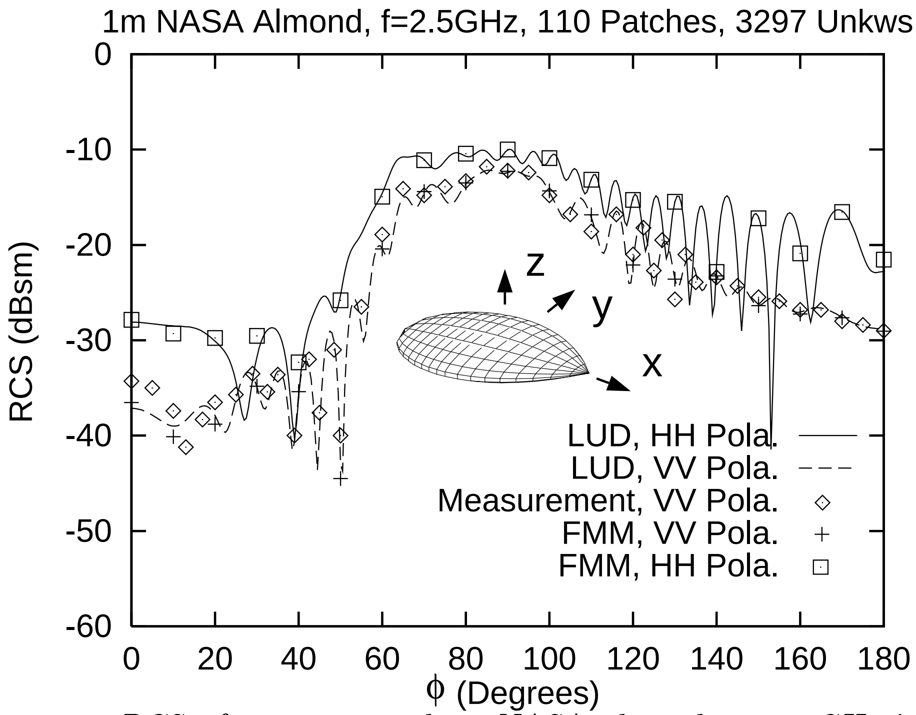

Figure 4. RCS of a one-meter long NASA almond at 2.5 GHz in the xy plane with $\theta = 90^o$. Five unknowns are used per wavelength. The results are computed with LU decomposition, and partially with FMM. The experimental measurement by Ohio State University [12] is also given for comparison.

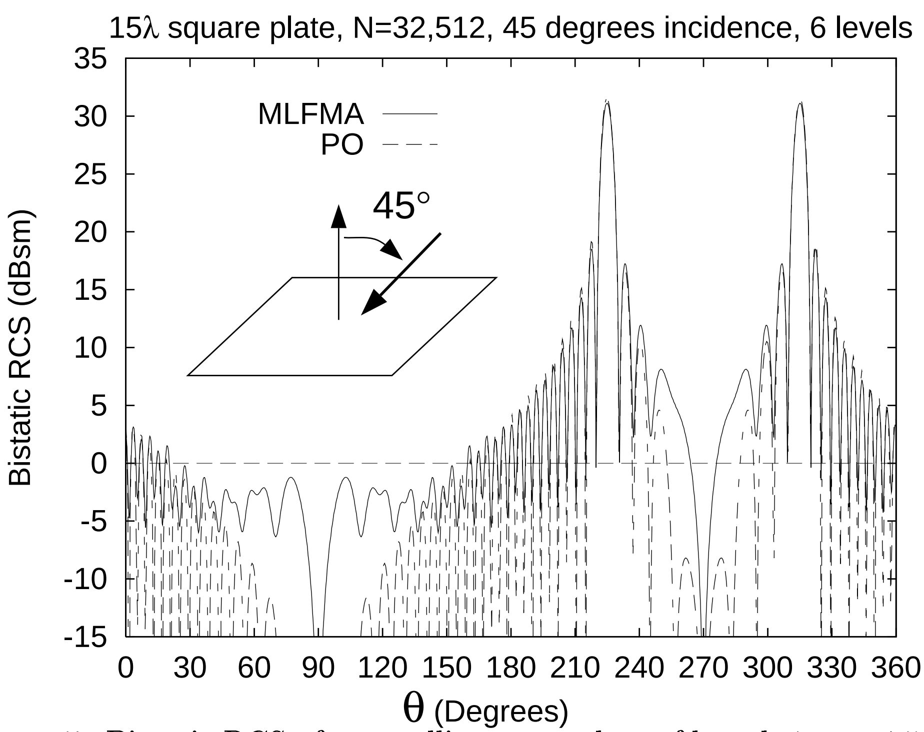

Figure 5. Bistatic RCS of a metallic square plate of length 1m at 4.5GHz for VV polarization with 45° incident angle.