Elastic Wave Lecture Notes, ECE471
University of Illinois at Urbana-Champaign

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Elastic Wave Class Note 1, ECE471, U. of Illinois

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Derivation of the Elastic Wave Equation (optional reading)

The elastic eave equation governs the propagation of the waves in solids. We shall illustrate its derivation as follows: the waves in a solid cause perturbation of the particles in the solid. The particles are displaced from their equilibrium position. The elasticity of the solid will provide the restoring force for the displaced particles. Hence, the study of the balance of these forces will lead to the elastic wave equation.

The displacement of the particles in a solid from their equilibrium position causes a *displacement field* $\mathbf{u}(\mathbf{x}, t)$ where $\mathbf{u}$ is the displacement of the particle at position $\mathbf{x}$ at time t. Here, $\mathbf{x}$ is a position vector in three dimensions. Usually, we use $\mathbf{r}$ for position vector, but we use $\mathbf{x}$ here so that indicial notation can be used conveniently. In indicial notation, x_1, x_2, and x_3 refer to x, y, z respectively. The displacement field $\mathbf{u}(\mathbf{x}, t)$ will stretch and compress distances between particles. For instance, particles at $\mathbf{x}$ and $\mathbf{x} + \delta\mathbf{x}$ are $\delta\mathbf{x}$ apart at equilibrium. But under a perturbation by $\mathbf{u}(\mathbf{x}, t)$, the charge in their separation is given by

$$\delta\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x} + \delta\mathbf{x}, t) - \mathbf{u}(\mathbf{x}, t) \tag{1}$$

By using Taylor's series expansion, the above becomes

$$\delta\mathbf{u}(\mathbf{x}, t) \simeq \delta\mathbf{x} \cdot \nabla\mathbf{u}(\mathbf{x}, t) + O(\delta\mathbf{x}^2) \tag{2}$$

or in indicial notation,

$$\delta u_i \simeq \partial_j u_i \delta x_j \tag{3}$$

where $\partial_j u_i = \frac{\partial u_i}{\partial x_j}$. This change in separation $\delta\mathbf{u}$ can be decomposed in to a symmetric and an antisymmetric part as follows:

$$\delta u_i = \frac{1}{2} \overbrace{(\partial_j u_i + \partial_i u_j)}^{symmetric} \delta x_j + \frac{1}{2} \overbrace{(\partial_j u_i - \partial_i u_j)}^{antisymmetric} \delta x_j \tag{4}$$

Using indicial notation, it can be shown that

$$\begin{aligned}
[(\nabla \times \mathbf{u}) \times \delta\mathbf{x}]_i &= \epsilon_{ijk}(\nabla \times \mathbf{u})_j \delta x_k \\
&= \epsilon_{ijk}\epsilon_{jlm}\partial_l u_m \delta x_k
\end{aligned} \tag{5}$$

From the identity that

$$\epsilon_{ijk}\epsilon_{jlm} = -\epsilon_{jik}\epsilon_{jlm} = \delta_{im}\delta_{kl} - \delta_{il}\delta_{km} \tag{6}$$

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we deduce that

$$[(\nabla \times \mathbf{u}) \times \delta\mathbf{x}]_i = (\partial_k u_i - \partial_i u_k)\delta x_k \tag{7}$$

Hence,

$$\delta u_i = \overbrace{e_{ij}\delta x_j}^{stretch} + \frac{1}{2}\overbrace{[(\nabla \times \mathbf{u}) \times \delta\mathbf{x}]_i}^{rotation} = (\bar{\mathbf{e}} \cdot \delta\mathbf{x})_i + \frac{1}{2}[\nabla \times \mathbf{u}) \times \delta\mathbf{x}]_i \tag{8}$$

where we have defined

$$e_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j)) \tag{9}$$

The first term in (8) results in a change in distance between the particles, while the second term, which corresponds to a rotation, has a higher order effect. This can be shown easily as follows: The perturbed distance between the particles at $\mathbf{x}$ and $\mathbf{x} + \delta\mathbf{x}$ is now $\delta\mathbf{x} + \delta\mathbf{u}$. The length square of this distance is (using (8))

$$\begin{aligned}(\delta\mathbf{x} + \delta\mathbf{u})^2 &\cong \delta\mathbf{x} \cdot \delta\mathbf{x} + 2\delta\mathbf{x} \cdot \delta\mathbf{u} + O(\delta\mathbf{u}^2)\\ &= |\delta\mathbf{x}|^2 + 2\delta\mathbf{x} \cdot \bar{\mathbf{e}} \cdot \delta\mathbf{x} + O(\delta\mathbf{u}^2)\end{aligned} \tag{10}$$

assuming that $\delta\mathbf{u} \ll \delta\mathbf{x}$, since $\mathbf{u}$ is small, and $\delta\mathbf{u}$ is even smaller. The second term in (8) vanished in (10) because it is orthogonal to $\delta\mathbf{x}$.

The above analysis shows that the stretch in the distance between the particles is determined to first order by the first term in (8). The tensor $\bar{\mathbf{e}}$ describes how the particles in a solid are stretched in the presence of a displacement field: it is called the *strain tensor*. This strain produced by the displacement field will produce stresses in the solid.

Stress in a solid is described by a *stress tensor* $\bar{\boldsymbol{\mathcal{T}}}$. Given a surface $\triangle S$ in the body of solid with a unit normal $\hat{n}$, the stress in the solid will exert a force on this surface $\triangle S$. This force acting on a surface, known as *traction*, is given by

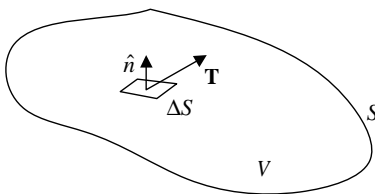$$\mathbf{T} = \hat{n} \cdot \bar{\boldsymbol{\mathcal{T}}}\triangle S \tag{11}$$

Figure 1:

Hence, if we know the traction on the surface S of a volume V, the total force acting on the body is given by

$$\oint_S \hat{n} \cdot \boldsymbol{\mathcal{T}} dS = \iiint_V \nabla \cdot \boldsymbol{\mathcal{T}} dV \tag{12}$$

where the second equality follows from Gauss' theorem, assuming that $\boldsymbol{\mathcal{T}}$ is defined as a continuous function of space. This force caused by stresses in the solid, must be

balanced by other forces acting on the body, e.g, the inertial force and body forces. Hence

$$\iiint_V \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} dV = \int_v \nabla \cdot \bar{\boldsymbol{\mathcal{T}}} dV + \int_V \mathbf{f} dV \tag{13}$$

where ρ is the mass density and $\mathbf{f}$ is a force density, e.g., due to some externally applied sources in the body. The left hand side is the inertial force while the right hand side is the total applied force on the body. Since (13) holds true for a arbitrary volume V, we have

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \nabla \cdot \bar{\boldsymbol{\mathcal{T}}} + \mathbf{f} \tag{14}$$

as our equation of motion.

Since the stress force, the first term on the right hand side of (14), is caused by strains in the solid, $\bar{\boldsymbol{\mathcal{T}}}$ should be a function of $\bar{\mathbf{e}}$. Under the assumption of small perturbation, $\boldsymbol{\mathcal{T}}$ should be linearly independent on $\bar{\mathbf{e}}$. The most general linear relationship between the second rank tensor is

$$\mathcal{T}_{ij} = C_{ijkl} e_{kl} \tag{15}$$

The above is the *constitutive relation* for a solid. C_{ijkl} is a fourth rank tensor.

For an isotropic medium, C_{ijkl} should be independent of any coordinate rotation. The most general form for a fourth rank tensor that is independent of coordinate rotation is [see Exercise 1]

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu_1 \delta_{ik} \delta_{jl} + \mu_2 \delta_{il} \delta_{jk} \tag{16}$$

Furthermore, since e_{kl} is symmetric, $C_{ijkl} = C_{ijlk}$. Therefore,

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu(\delta_{jk} \delta_{il} + \delta_{il} \delta_{ik}) \tag{17}$$

Consequently, in an isotropic medium, the constitutive relation is characterized by two constants λ and μ known as Lamé constant. Note that as a consequence of (17), $\mathcal{T}_{ij} = \mathcal{T}_{ji}$ in (15). Hence, both the strain and the stress tensors are symmetric tensors.

Using (17) in (15), we have after using (9) that

$$\begin{aligned} \mathcal{T}_{ij} &= \lambda \delta_{ij} e_{ll} + \mu(e_{ij} + e_{ji}) \\ &= \lambda \delta_{ij} \partial_l u_l + \mu(\partial_j u_i + \partial_i u_j). \end{aligned} \tag{18}$$

Then

$$\begin{aligned} (\nabla \cdot \bar{\boldsymbol{\mathcal{T}}})_j &= \partial_i \mathcal{T}_{ij} = \partial_j \lambda \partial_l u_l + \partial_i \mu \partial_j u_i + \partial_i \mu \partial_i u_j \\ &= \lambda \partial_i \partial_l u_l + (\partial_l u_l) \partial_j \lambda + \mu \partial_j \partial_i u_i + (\partial_j u_i) \partial_i \mu + \partial_i \mu \partial_i u_j \\ &= (\lambda + \mu)[\nabla \nabla \cdot \mathbf{u}]_j + (\nabla \cdot \mu \nabla \mathbf{u})_j + (\nabla \cdot \mathbf{u})(\nabla \lambda)_j + [(\nabla \mathbf{u}) \cdot \nabla \mu]_j \end{aligned} \tag{19}$$

If μ and λ are constants of positions, we have

$$\nabla \cdot \bar{\boldsymbol{\mathcal{T}}} = (\lambda + \mu) \nabla \nabla \cdot \mathbf{u} + \mu \nabla^2 \mathbf{u} \tag{20}$$

Using (20) in (14), we have

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + \mu) \nabla \nabla \cdot \mathbf{u} + \mu \nabla^2 \mathbf{u} + \mathbf{f} \tag{21}$$

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which is the elastic wave equation for homogeneous and isotropic media.

If μ, which is also the shear modulus, is zero, taking the divergence of (21), and defining $\theta = \nabla \cdot \mathbf{u}$, we have

$$\rho \frac{\partial^2 \theta}{\partial \mathcal{T}^2} = \lambda \nabla^2 \theta + \nabla \cdot \mathbf{f} \tag{22}$$

which is the acoustic wave equation for homogenous media, and λ is the bulk modulus.

Exercise 1

1.(a) Under coordinate rotations, show that the fourth rank tensor C_{ijkl} transforms as

$$C_{ijkl} = T_{ii'} T_{jj'} T_{kk'} T_{ll'} C'_{i'j'k'l'}$$

(b) Show that if

$$C'_{i'j'k'l'} = \lambda_1 \delta_{i'j'} \delta_{k'l'} + \mu_1 \delta_{i'k'} \delta_{j'l'} + \mu_2 \delta_{i'l'} \delta_{j'k'}$$

then under coordinate rotation,

$$C_{ijkl} = \lambda_1 \delta_{ij} \delta_{kl} + \mu_1 \delta_{ik} \delta_{jl} + \mu_2 \delta_{il} \delta_{jk}$$

In other words, C_{ijkl} remains the same under coordinate rotation, i.e., it is an isotropic tensor.

(c) Proof that the form in (b) is the only form for isotropic fourth rank tensor (difficult).

References

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[3] Archenbach, J. D.,*Wave Propagation in Elastic Solids*, North Holland, Amsterdam, 1973.

Elastic Wave Class Note 2, ECE471, U. of Illinois

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Solution of the Elastic Wave Equation—A Succinct Derivation

The elastic wave equation is

$$(\lambda + 2\mu)\nabla\nabla \cdot \mathbf{u} - \mu\nabla \times (\nabla \times \mathbf{u}) - \rho\ddot{\mathbf{u}} = -\mathbf{f}(\mathbf{x}, t). \tag{1}$$

By Fourier transform,

$$\mathbf{u}(\mathbf{x}, t), = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{-i\omega t}\mathbf{u}(\mathbf{x}, \omega) \tag{2}$$

(1) becomes

$$(\lambda + 2\mu)\nabla\nabla \cdot \mathbf{u} - \mu\nabla \times (\nabla \times \mathbf{u}) + \omega^2\rho\mathbf{u} = -\mathbf{f} \tag{3}$$

where $\mathbf{u} = \mathbf{u}(\mathbf{x}, \omega)$, and $\mathbf{f} = \mathbf{f}(\mathbf{x}, \omega)$ now. By taking $\nabla\times$ of the above equation, and defining $\mathbf{\Omega} = \nabla \times \mathbf{u}$, the rotation of $\mathbf{u}$, we have

$$\mu\nabla \times (\nabla \times \mathbf{\Omega}) - \omega^2\rho\mathbf{\Omega} = \nabla \times \mathbf{f} \tag{4}$$

Since $\nabla \cdot \mathbf{\Omega} = \nabla \cdot \nabla \times \mathbf{u} = 0$, the above is just

$$\nabla^2\mathbf{\Omega} + k_s^2\mathbf{\Omega} = -\frac{1}{\mu}\nabla \times \mathbf{f} \tag{5}$$

where $k_s^2 = \omega^2\rho/\mu = \omega^2/c_s^2$ and $c_s = \sqrt{\frac{\mu}{\rho}}$. The solution to the above equation is

$$\mathbf{\Omega}(\mathbf{x}, \omega) = \frac{1}{\mu}\int d\mathbf{x}' g_s(\mathbf{x} - \mathbf{x}')\nabla' \times \mathbf{f}\left(\mathbf{x}', \omega\right) \tag{6}$$

where $g_s(\mathbf{x} - \mathbf{x}') = e^{ik_s|\mathbf{x}-\mathbf{x}'|}/4\pi|\mathbf{x} - \mathbf{x}'|$. Taking the divergence of (3), and defining $\theta = \nabla \cdot \mathbf{u}$, the dilational part of $\mathbf{u}$, we have

$$(\lambda + 2\mu)\nabla^2\theta + \omega^2\rho\theta = -\nabla \cdot \mathbf{f}. \tag{7}$$

The solution to the above is

$$\theta(\mathbf{x}, \omega) = \frac{1}{\lambda + 2\mu}\int d\mathbf{x}' g_c(\mathbf{x} - \mathbf{x}')\nabla' \cdot \mathbf{f}(\mathbf{f}\left(\mathbf{x}', \omega\right) \tag{8}$$

where $k_c^2 = \omega^2\rho/(\lambda + 2\mu) = \omega^2/c_c^2$, $c_c = \sqrt{(\lambda + 2\mu)/\rho}$, and $g_c(\mathbf{x} - \mathbf{x}') = e^{ik_c|\mathbf{x}-\mathbf{x}'|}/4\pi|\mathbf{x} - \mathbf{x}'|$.

From (3), we deduce that

$$\mathbf{u}(\mathbf{x}, \omega) = -\frac{\mathbf{f}}{\mu k_s^2} + \frac{1}{k_s^2}\nabla \times \Omega - \frac{1}{k_c^2}\nabla\theta \tag{9}$$

Then, using (6) and (8), we have

$$\mathbf{u}(\mathbf{x}, \omega) = -\frac{\mathbf{f}}{\mu k_s^2} + \frac{1}{\mu k_s^2}\nabla \times \int d\mathbf{x}' g_s(\mathbf{x} - \mathbf{x}')\nabla' \times \mathbf{f}(\mathbf{x}', \omega) -$$
$$\frac{1}{(\lambda + 2\mu)k_c^2}\nabla \int d\mathbf{x}' g_c(\mathbf{x} - \mathbf{x}')\nabla' \cdot \mathbf{f}(\mathbf{x}', \omega) \tag{10}$$

Using integration by parts, and the fact that $\nabla' g(\mathbf{x} - \mathbf{x}') = -\nabla g(\mathbf{x} - \mathbf{x}')$, the above

$$\mathbf{u}(\mathbf{x}, \omega) = -\frac{\mathbf{f}}{\mu k_s^2} + \frac{1}{\mu k_s^2}\nabla \times \nabla \times \int d\mathbf{x}' g_s(\mathbf{x} - \mathbf{x}')\mathbf{f}(\mathbf{x}', \omega)$$
$$- \frac{1}{(\lambda + 2\mu)k_c^2}\nabla\nabla \int d\mathbf{x}' g_c(\mathbf{x} - \mathbf{x}')\mathbf{f}(\mathbf{x}', \omega) \tag{11}$$

Using $\nabla \times \nabla \times \mathbf{A} = (\nabla\nabla - \nabla^2)\mathbf{A}$, and the fact that $\nabla^2 g_s((\mathbf{x} - \mathbf{x}') - \delta(\mathbf{x} - \mathbf{x}')$, the above becomes

$$\mathbf{u}(\mathbf{x}, \omega) = \left(\bar{\mathbf{I}} + \frac{\nabla\nabla}{k_s^2}\right) \cdot \frac{1}{\mu}\int d\mathbf{x}' g_s(\mathbf{x} - \mathbf{x}')\mathbf{f}(\mathbf{x}', \omega)$$
$$- \frac{\nabla\nabla}{k_c^2} \cdot \frac{1}{\lambda + 2\mu}\int d\mathbf{x}' g_c(\mathbf{x} - \mathbf{x}')\mathbf{f}(\mathbf{x}', \omega) \tag{12}$$

Time-Domain Solution

If $\mathbf{f}(\mathbf{x}, t) = \hat{x}_j\delta(\mathbf{x})\delta(t)$, then $\mathbf{f}(\mathbf{x}, \omega) = \hat{x}_j\delta(\mathbf{x})$ and

$$\mathbf{u}(\mathbf{x}, \omega) = \left(\bar{\mathbf{I}} + \frac{\nabla\nabla}{k_s^2}\right) \cdot \frac{e^{ik_s r}}{4\pi\mu r}\hat{x}_j - \frac{\nabla\nabla}{k_c^2} \cdot \frac{e^{ik_c r}}{4\pi(\lambda + 2\mu)r}\hat{x}_j \tag{13}$$

or in indicial notation

$$u_i(\mathbf{x}, \omega) = \delta_{ij}\frac{e^{ik_s r}}{4\pi\mu r} + \partial_i\partial_j\frac{1}{4\pi r}\left(\frac{e^{ik_s r}}{\mu k_s^2} - \frac{e^{ik_c r}}{(\lambda + 2\mu)k_c^2}\right) \tag{14}$$

Since $k_s = \omega/c_s = \omega\sqrt{\rho/\mu}$, $k_c = \omega/c_c = \omega\sqrt{\rho/(\lambda + 2\mu)}$, the above can be inverse Fourier transformed to yield

$$u_i(\mathbf{x}, t) = \delta_{ij}\frac{\delta(t - r/c_s)}{4\pi\mu r} - \partial_i\partial_j\frac{1}{4\pi r\rho}\left[\left(t - \frac{r}{c_s}\right)u\left(t - \frac{r}{c_s}\right) - \left(t - \frac{r}{c_c}\right)u\left(t - \frac{r}{c_c}\right)\right] \tag{15}$$

Writing $\partial_i\partial_j = (\partial_i r)(\partial_j r)\frac{\partial^2}{\partial r^2}$, the above becomes

$$u_i(\mathbf{x}, t) = [\delta_{ij} - (\partial_i r)(\partial_j r)]\frac{\delta(t - r/c_s)}{4\pi\mu r} - (\partial_i r)(\partial_j r)\frac{1}{2\pi\rho r^3}\left[tu\left(t - \frac{r}{c_s}\right) - tu\left(t - \frac{r}{c_c}\right)\right]$$
$$+ (\partial_i r)(\partial_j r)\frac{\delta(t - r/c_c)}{4\pi(\lambda + 2\mu)r}, \tag{16}$$

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Consequently, if $\mathbf{f}(\mathbf{x}, t)$ is a general body force, by convolutional theorem,

$$
\begin{aligned}
u_i(\mathbf{x}, t) = {} & [\delta_{ij} - (\partial_i r)(\partial_j r)] \frac{f_j(\mathbf{x}, t - \frac{r}{c_s})}{4\pi\mu r} \\
& + (\partial_i r)(\partial_j r) \frac{1}{2\pi\rho r^3} \int_{r/c_c}^{r/c_s} \tau f_j(\mathbf{x}, t - \tau) d\tau \\
& + (\partial_i r)(\partial_j r) \frac{f_j(\mathbf{x}, t - \frac{r}{c_c})}{4\pi(\lambda + 2\mu)r}
\end{aligned}
\tag{17}
$$

Solution of the Elastic Wave Equation—Fourier-Laplace Transform

The elastic wave equation

$$
(\lambda + \mu)\nabla\nabla \cdot \mathbf{u} + \mu\nabla^2\mathbf{u} - \rho\ddot{\mathbf{u}} = -\mathbf{f}
\tag{18}
$$

can be solved by Fourier-Laplace transform. We let

$$
\mathbf{u}(\mathbf{x}, t) = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \int_{-\infty}^{\infty} d\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}} \mathbf{u}(\mathbf{k}, \omega),
\tag{19}
$$

then (18) becomes

$$
-(\lambda + \mu)\mathbf{k}\mathbf{k} \cdot \mathbf{u} - \mu k^2\mathbf{u} + \omega^2\rho\mathbf{u} = -\mathbf{f}
\tag{20}
$$

where $\mathbf{u} = \mathbf{u}(\mathbf{k}, \omega)$, $\mathbf{f} = \mathbf{f}(\mathbf{k}, \omega)$ now. The above can be formally solved to yield

$$
\mathbf{u}(\mathbf{k}, \omega) = \left[(\lambda + \mu)\mathbf{k}\mathbf{k} + \mu k^2\bar{\mathbf{I}} - \omega^2\rho\bar{\mathbf{I}}\right]^{-1} \cdot \mathbf{f}.
\tag{21}
$$

The inverse of the above tensor commutes with itself, so that the inverse must be of the form $\alpha\bar{\mathbf{I}} + \beta\mathbf{k}\mathbf{k}$, i.e.,

$$
\left[(\lambda + \mu)\mathbf{k}\mathbf{k} + \mu k^2\bar{\mathbf{I}} - \omega^2\rho\bar{\mathbf{I}}\right] \cdot \left[\alpha\bar{\mathbf{I}} + \beta\mathbf{k}\mathbf{k}\right] = \bar{\mathbf{I}}
\tag{22}
$$

The above yields that

$$
\alpha = \frac{1}{(\mu k^2 - \omega^2\rho)}
\tag{23}
$$

$$
\begin{aligned}
\beta &= \frac{-(\lambda + \mu)}{\left[\mu k^2 - \rho\omega^2\right]\left[(\lambda + 2\mu)k^2 - \rho\omega^2\right]} \\
&= \frac{\mu}{\rho\omega^2\left[\mu k^2 - \rho\omega^2\right]} + \frac{(\lambda + 2\mu)}{\rho\omega^2\left[(\lambda + 2\mu)k^2 - \rho\omega^2\right]} \\
&= -\frac{1}{k_s^2\mu\left[k^2 - k_s^2\right]} + \frac{1}{k_c^2(\lambda + 2\mu)\left[k^2 - k_c^2\right]}
\end{aligned}
\tag{24}
$$

where $k_s^2 = \omega^2\rho/\mu$, $k_c^2 = \omega^2\rho/(\lambda + 2\mu)$. Consequently,

$$
\mathbf{u}(\mathbf{k}, \omega) = \left[\bar{\mathbf{I}} - \frac{\mathbf{k}\mathbf{k}}{k_s^2}\right] \cdot \frac{\mathbf{f}(\mathbf{k}, \omega)}{\mu(k^2 - k_s^2)} + \frac{\mathbf{k}\mathbf{k} \cdot \mathbf{f}(\mathbf{k}, \omega)}{k_c^2(\lambda + 2\mu)(k^2 - k_c^2)}
\tag{25}
$$

It can be shown that

$$
\int d\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}} \left(\bar{\mathbf{I}} - \frac{\mathbf{k}\mathbf{k}}{k_s^2}\right) \frac{1}{\mu(k^2 - k_s^2)} = \left(\bar{\mathbf{I}} + \frac{\nabla\nabla}{k_s^2}\right) \frac{e^{ik_s r}}{4\pi\mu r}
\tag{26}
$$

3

where $k_s = \omega/c_s$, $c_s = \sqrt{\mu/\rho}$ is the shear velocity, and $r = |\mathbf{x}|$. Similarly,

$$\int d\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}} \frac{\mathbf{k}\mathbf{k}}{k_c^2(\lambda + 2\mu)(k^2 - k_c^2)} = -\frac{\nabla\nabla}{k_c^2} \frac{e^{ik_c r}}{4\pi(\lambda + 2\mu)r} \tag{27}$$

where $k_c = \omega/C_c$, $C_c = \sqrt{(\lambda + 2\mu)/\rho}$ is the compressed velocity.

By convolutional theorem,

$$\begin{aligned}
\mathbf{u}(\mathbf{x}, \omega) &= \left(\bar{\mathbf{I}} + \frac{\nabla\nabla}{k_s^2}\right) \cdot \int d\mathbf{x}' \frac{e^{ik_s|\mathbf{x}-\mathbf{x}'|}}{4\pi\mu|\mathbf{x} - \mathbf{x}'|} \mathbf{f}(\mathbf{x}', \omega) \\
&\quad - \frac{\nabla\nabla}{k_c^2} \cdot \int d\mathbf{x}' \frac{e^{ik_c|\mathbf{x}-\mathbf{x}'|}}{4\pi(\lambda + 2\mu)|\mathbf{x} - \mathbf{x}'|} \mathbf{f}(\mathbf{x}', \omega)
\end{aligned} \tag{28}$$

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Boundary Conditions for Elastic Wave Equation

The equation of motion for elastic waves is

$$\nabla \cdot \bar{\mathcal{T}} + \mathbf{f} = -\omega^2 \rho \ddot{\mathbf{u}} \tag{1}$$
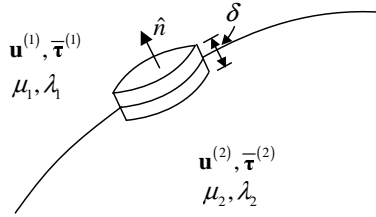

Figure 1:

By integrating this over a pill-box whose thickness is infinitesimally small at the interface between two regions, and assuming that $\mathbf{f}$ is not singular at the interface, it can be shown that

$$\hat{n} \cdot \bar{\boldsymbol{\mathcal{T}}}^{(1)} = \hat{n} \cdot \bar{\boldsymbol{\mathcal{T}}}^{(2)} \tag{2}$$

The above corresponds to three equations for the boundary conditions at an interface.

If (2) is written in terms of Cartesian coordinates with $\hat{z}$ being the unit normal $\hat{n}$, then (2) is equivalent to $\mathcal{T}_{zz}$, $\mathcal{T}_{zx}$, and $\mathcal{T}_{zy}$ continuous across an interface. Since

$$\mathcal{T}_{zz} = \lambda \nabla \cdot \mathbf{u} + 2\mu \partial_z u_z = \lambda(\partial_x u_x + \partial_y u_y) + (\lambda + 2\mu)\partial_z u_z \tag{3}$$

and $\mathcal{T}_{zz}$ cannot be singular (does not contain Dirac delta functions), then $\partial_x u_x$, $\partial_y u_y$ and $\partial_z u_z$ must be regular. Requiring $\partial_z u_z$ to be regular implies that at the interface

$$u_z^{(1)} = u_z^{(2)} \tag{4}$$

Furthermore,

$$\mathcal{T}_{zx} = \mu \partial_x u_z + \mu \partial_z u_x, \tag{5}$$

the continuity of $\mathcal{T}_{zx}$ implies that $\partial_z u_x$ must be regular. This induces the boundary condition that

$$u_x^{(1)} = u_x^{(2)} \tag{6}$$

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By the same argument from $\mathcal{T}_{zy}$, we have

$$u_y^{(1)} = u_y^{(2)} \tag{7}$$

Hence, in addition to (2), we have boundary conditions (4), (6), and (7) which form a total of 6 boundary conditions at a solid-solid interface.

At a fluid-solid interface, $\mu_1 = 0$ in one region, then $\mathcal{T}_{zx}$ and $\mathcal{T}_{zy}$ are zero at the interface, in order for them to be continuous across an interface. Further, u_x and u_y need not be continuous anymore. The boundary conditions are (2) and (4), a total of 4 boundary conditions.

At a fluid-fluid interface, only $\mathcal{T}_{zz}$ is nonzero. Its continuity implies $\lambda \nabla \cdot \mathbf{u} = p$ is continuous or the pressure is continuous. Furthermore, it induces the boundary condition (4). Hence there are only 2 boundary conditions.

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Elastic Wave Equation for Planarly Layered Media

The elastic wave equation for isotropic inhomogeneous media is

$$\partial_j(\lambda \partial_l u_l) + \partial_i(\mu \partial_i u_j) + \partial_i(\mu \partial_j u_i) + \omega^2 \rho u_j = 0 \tag{1}$$

In vector notation, this may be written as

$$\nabla(\lambda \nabla \cdot \mathbf{u}) + \nabla \cdot (\mu \nabla \mathbf{u}) + (\mu \nabla \mathbf{u}) \cdot \overleftarrow{\nabla} + \omega^2 \rho \mathbf{u} = 0 \tag{2}$$

where $\overleftarrow{\nabla}$ operates on terms to its left.

If λ and μ are functions of z only, and $\frac{\partial}{\partial x} = 0$, and $\mathbf{u} = \hat{x} u_x$, then extracting the $\hat{x}$ component of (2), we have

$$\nabla_s \cdot \mu \nabla_s u_x + \omega^2 \rho u_x = 0, \tag{3}$$

where $\nabla_s = \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$.
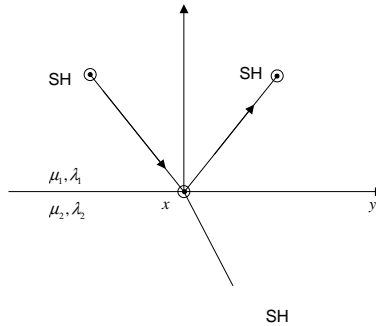

Figure 1:

Hence for this problem, a displacement field polarized in x with no variation in x is a pure shear wave. For instance, an SH (shear horizontal) plane wave will have this property. Even when μ and λ are discontinuous in z, only SH waves will be reflected and transmitted.

However, if the incident plane wave is an SV (shear vertical) wave, the displacement of the particles at an interface will induce both P (compressional) and SV reflected and transmitted waves. To see this , we let $\mathbf{u} = \hat{y} u_y + \hat{z} u_z$. Then (2) becomes

$$\nabla_s(\lambda \nabla_s \cdot \mathbf{u}_s) + \nabla_s \cdot (\mu \nabla_s \mathbf{u}_s) + (\mu \nabla_s \mathbf{u}_s) \cdot \overleftarrow{\nabla}_s + \omega^2 \rho \bar{u}_s = 0. \tag{4}$$

The above is the equation that governs the shear and compressional waves in a one-dimensional inhomogeneity where $\frac{\partial}{\partial x} = 0$.

Since λ, μ and $\mathbf{u}_s$ are smooth functions of y, $\frac{\partial}{\partial y}$ is smooth. In order for $\frac{\partial}{\partial z}$ to be nonsingular in (4), we need that

$$\lambda \nabla_s \cdot \mathbf{u}_s + 2\mu \frac{\partial}{\partial z} u_z \tag{5}$$

be continuous. This is the same as requiring $\mathcal{T}_{zz}$ be continuous.

Similarly, taking the $\hat{y}$ component of (4) that contains $\frac{\partial}{\partial z}$ derivatives, we require that

$$\mu \frac{\partial}{\partial z} u_y + \mu \frac{\partial}{\partial y} u_z \tag{6}$$

to be continuous. This is the same as requiring $\mathcal{T}_{zy}$ to be continuous. In order for (5) and 6) to be regular, u_z and u_y have to be continuous functions of z. Hence, the boundary conditions at an Solid-solid interface are

$$u_z^{(1)} = u_z^{(2)}$$

$$u_y^{(1)} = u_y^{(2)} \tag{7b}$$

$$\lambda_1 \nabla_s \cdot \mathbf{u}_s^{(1)} + 2\mu_1 \frac{\partial}{\partial z} u_z^{(1)} = \lambda_2 \nabla_s \cdot \mathbf{u}_s^{(2)} + 2\mu_2 \frac{\partial}{\partial z} u_z^{(2)} \tag{7c}$$

$$\mu_1 \left(\frac{\partial}{\partial z} u_y^{(1)} + \frac{\partial}{\partial y} u_z^{(1)} \right) = \mu_2 \left(\frac{\partial}{\partial z} u_y^{(2)} + \frac{\partial}{\partial y} u_z^{(2)} \right) \tag{7d}$$

The reflection of SH waves by a plane interface is purely an Scalar problem. However, the reflection of a P wave or an SV wave by a plane boundary is a vector problem. In this case $\mathbf{u}_s$ can always be decomposed into two components $\mathbf{u}_s = \hat{v} u_v + \hat{p} u_p$ where $\hat{p}$ is a unit vector in the direction of wave propagation, and $\hat{v}$ is a unit vector in the yz-plane orthogonal to $\hat{p}$.

For $z > 0$, the incident wave can be written as

$$\mathbf{u}_s^{ins} = \begin{bmatrix} u_v^{inc} \\ u_p^{inc} \end{bmatrix} = \begin{bmatrix} v_0 e^{-ik_{1vz}z} \\ p_0 e^{-ik_{1pz}z} \end{bmatrix} e^{ik_y y} = \begin{bmatrix} e^{-ik_{1vz}z} & 0 \\ 0 & e^{-ik_{1pz}z} \end{bmatrix} \begin{bmatrix} v_0 \\ p_0 \end{bmatrix} e^{ik_y y}$$

$$= e^{-i\bar{\mathbf{k}}_{1z}z} \cdot \mathbf{u}_0 e^{ik_y y} \tag{8}$$

In the presence of a boundary , the reflected wave can be written as

$$\mathbf{u}_s^{ref} = \begin{bmatrix} u_v^{ref} \\ u_p^{ref} \end{bmatrix} = e^{i\bar{\mathbf{k}}_{1z}z} \cdot \mathbf{u}_r e^{ik_y y} \tag{9}$$

The most general relation between $\mathbf{u}_r$ and $\mathbf{u}_0$ is that

$$\mathbf{u}_r = \bar{\mathbf{R}} \cdot \mathbf{u}_0 \tag{10}$$

where

$$\bar{\mathbf{R}} = \begin{bmatrix} R_{vv} & R_{vp} \\ R_{pv} & R_{pp} \end{bmatrix} \tag{11}$$

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By the same token, the transmitted wave is

$$\mathbf{u}_s^{tra} = e^{-i\bar{\mathbf{k}}_{2z}z} \cdot \mathbf{u}_t e^{ik_y y} \tag{12}$$

where

$$\mathbf{u}_t = \bar{\mathbf{T}} \cdot \mathbf{u}_0 \tag{13}$$

and

$$\bar{\mathbf{T}} = \begin{bmatrix} T_{vv} & T_{vp} \\ T_{pv} & T_{pp} \end{bmatrix} \tag{12a}$$

There are 4 unknowns in $\mathbf{u}_r$ and $\mathbf{u}_t$ which can be found from 4 equations as a consequence of (7a) to (7d).
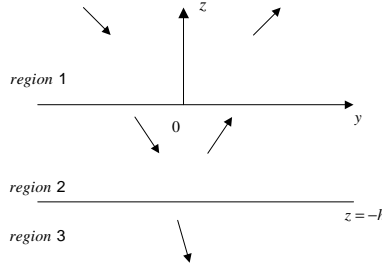

Figure 2:

When three regions are present as shown above, the field in Region 1 can be written as

$$\begin{aligned} \mathbf{u}_1 &= e^{-i\bar{\mathbf{k}}_{1z}z} \cdot \mathbf{a}_1 + e^{i\bar{\mathbf{k}}_{1z}z} \cdot \mathbf{b}_1, \\ &= \left[e^{-i\bar{\mathbf{k}}_{1z}z} + e^{i\bar{\mathbf{k}}_{1z}z} \cdot \tilde{\bar{\mathbf{R}}}_{12} \right] \cdot \mathbf{a}_1 \end{aligned} \tag{14}$$

where we have defined $\mathbf{b}_1 = \tilde{\bar{\mathbf{R}}}_{12} \cdot \mathbf{a}_1$. The $e^{ik_y y}$ dependance is dropped assuming that it is implicit.

In Region 2, we have

$$\begin{aligned} \mathbf{u}_2 &= e^{-i\bar{\mathbf{k}}_{2z}z} \cdot \mathbf{a}_2 + e^{i\bar{\mathbf{k}}_{2z}z} \cdot \mathbf{b}_2 \\ &= \left[e^{-i\bar{\mathbf{k}}_{2z}z} + e^{i\bar{\mathbf{k}}_{2z}(z+h)} \cdot \bar{\mathbf{R}}_{23} \cdot e^{i\bar{\mathbf{k}}_{2z}h} \right] \cdot \mathbf{a}_2 \end{aligned} \tag{15}$$

where we have defined

$$e^{-i\bar{\mathbf{k}}_{2z}h} \cdot \mathbf{b}_2 = \bar{\mathbf{R}}_{23} \cdot e^{i\bar{\mathbf{k}}_{2z}h} \cdot \mathbf{a}_2 \tag{14a}$$

and $\bar{\mathbf{R}}_{23}$ is just the one-interface reflection coefficient previously defined.

The amplitude $\mathbf{a}_2$ is determined by the transmission of the amplitude of the downgoing wave in Region 1 (which is a_1) plus the reflection of the upgoing wave in Region 2.

As a result, we have it at $z = 0$,

$$\mathbf{a}_2 = \bar{\mathbf{T}}_{12} \cdot \mathbf{a}_1 + \bar{\mathbf{R}}_{21} \cdot e^{i\bar{\mathbf{k}}_{2z}h} \cdot \bar{\mathbf{R}}_{23} \cdot e^{i\bar{\mathbf{k}}_{2z}h} \cdot \mathbf{a}_2 \tag{16}$$

The above solves to yield

$$\mathbf{a}_2 = \left[\bar{\mathbf{I}} - \bar{\mathbf{R}}_{21} \cdot e^{i\bar{\mathbf{k}}_{2z}h} \cdot \bar{\mathbf{R}}_{23} \cdot e^{i\bar{\mathbf{k}}_{2z}h} \right]^{-1} \cdot \bar{\mathbf{T}}_{12} \cdot \mathbf{a}_1 \tag{17}$$

The amplitude $\mathbf{b}_1$ of the upgoing wave in Region 1 is the consequence of the reflection of the downgoing wave in Region 1 plus the transmission of the upgoing wave in Region 2. Hence, at $z = 0$,

$$\mathbf{b}_1 = \widetilde{\bar{\mathbf{R}}}_{12} \cdot \mathbf{a}_1 = \bar{\mathbf{R}}_{12} \cdot \mathbf{a}_1 + \bar{\mathbf{T}}_{21} \cdot e^{i\bar{\mathbf{k}}_{2z}h} \cdot \bar{\mathbf{R}}_{23} \cdot e^{i\bar{\mathbf{k}}_{2z}h} \cdot \mathbf{a}_2 \qquad (18)$$

Using (16), the above can be solved for $\widetilde{\bar{\mathbf{R}}}_{12}$, yielding

$$\widetilde{\bar{\mathbf{R}}}_{12} = \bar{\mathbf{R}}_{12} \quad + \quad \bar{\mathbf{T}}_{12} \cdot e^{i\bar{\mathbf{k}}_{2z}h} \cdot \bar{\mathbf{R}}_{23} \cdot e^{i\bar{\mathbf{k}}_{2z}h}$$
$$\cdot \left[\bar{\mathbf{I}} - \bar{\mathbf{R}}_{21} \cdot e^{i\bar{\mathbf{k}}_{2z}h} \cdot \bar{\mathbf{R}}_{23} \cdot e^{i\bar{\mathbf{k}}_{2z}h} \right]^{-1} \cdot \bar{\mathbf{T}}_{12}, \qquad (19)$$

where $\widetilde{\bar{\mathbf{R}}}_{12}$ is the generalized reflection operator for a layered medium. If a region is added beyond Region 3, we need only to change $\bar{\mathbf{R}}_{23}$ to $\widetilde{\bar{\mathbf{R}}}_{23}$ in the above to account for subsurface reflection.

The above is a recursive relation which in general, can be written as

$$\widetilde{\bar{\mathbf{R}}}_{i,i+1} \quad = \quad \bar{\mathbf{R}}_{i,i+1} + \bar{\mathbf{T}}_{i+1,i} \cdot e^{i\bar{\mathbf{k}}_{i+1,z}h_{i+1}} \cdot \widetilde{\bar{\mathbf{R}}}_{i+1,i+2} \cdot e^{i\bar{\mathbf{k}}_{i+1,z}h} \cdot$$
$$\left[\bar{\mathbf{I}} - \bar{\mathbf{R}}_{i+1,i} \cdot e^{i\bar{\mathbf{k}}_{i+1,z}h_{i+1}} \cdot \widetilde{\bar{\mathbf{R}}}_{i+1,i+2} \cdot e^{i\bar{\mathbf{k}}_{i+1,z}h_{i+1}} \right]^{-1} \cdot \bar{\mathbf{T}}_{i,i+1} \qquad (20)$$

where h_{i+1} is the thickness of the $(i+1)$-th layer. Equation (17) is then

$$\mathbf{a}_{i+1} = \left[\bar{\mathbf{I}} - \bar{\mathbf{R}}_{i+1,i} \cdot e^{i\bar{\mathbf{k}}_{i+1,z}h} \cdot \widetilde{\bar{\mathbf{R}}}_{i+1,i+2} \cdot e^{i\bar{\mathbf{k}}_{i+1,z}h} \right]^{-1} \cdot \bar{\mathbf{T}}_{i,i+1} \cdot \mathbf{a}_i \qquad (21)$$

Elastic Wave Class Note 5, ECE471, U. of Illinois

W. C. Chew

Fall, 1991

Decomposition of Elastic Wave into SH, SV and P Waves

The preceding discussion shows that for elastic plane waves, the SH waves propagate through a planarly layered medium independent of the SV and P waves. Moreover, the SV and P waves are coupled together at the planar interfaces. Given an arbitrary source, we can use the Weyl or Sommerfeld identity to expand the waves into plane waves. If these plane waves can be further decomposed into SH, SV and P waves, then the transmission and reflection of these waves through a planarly layered medium can be easily found.

It has been shown previously that an arbitrary source produces a displacement field in a homogeneous isotropic medium given by

$$\mathbf{u}(\mathbf{r}) = -\frac{\mathbf{f}}{\mu k_s^2} + \underbrace{\frac{1}{k_s^2}\nabla \times \boldsymbol{\Omega}}_{\mathbf{u}^s} - \underbrace{\frac{1}{k_c^2}\nabla\theta}_{\mathbf{u}^p} \tag{1}$$

Outside the source region the first term is zero. The second term corresponds to S waves while the third term correspond to P waves. Hence, in a source-free region, the S component of (1) is

$$\mathbf{u}^s(\mathbf{r}) = \frac{1}{k_s^2}\nabla \times \boldsymbol{\Omega} \tag{2}$$

From the definition of $\boldsymbol{\Omega}$, we have

$$\boldsymbol{\Omega} = \nabla \times \mathbf{u}^s(\mathbf{r}). \tag{3}$$

In the above, $\boldsymbol{\Omega}$ was previously derived in Class Notes 2 to be

$$\boldsymbol{\Omega}(\mathbf{r}) = \frac{1}{\mu}\nabla \times \int d\mathbf{r}' \frac{e^{ik_s|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}\mathbf{f}(\mathbf{r}') \tag{4}$$

using the Sommerfeld-Weyl identities, (4) can be expressed as a linear superposition of plane waves. Assuming that $\boldsymbol{\Omega}$ and $\mathbf{u}^s$ are plane waves in (2) and (3), replacing ∇ by $i\mathbf{k}_s$, we note only the SV waves have $\mathbf{u}_z^s \neq 0$, and only the SH waves have $\Omega_z \neq 0$. Hence, we can use $\mathbf{u}_z^s$ to characterize SV waves and $\boldsymbol{\Omega}_z$ to characterize SH wave.

Assuming that $\mathbf{f}(\mathbf{r}) = \hat{a}A\delta(\mathbf{r})$, i.e., a point excitation polarized in the $\hat{a}$ direction, then (4) becomes

$$\boldsymbol{\Omega}(\mathbf{r}) = \frac{A}{\mu}(\nabla \times \hat{a})\frac{e^{ik_s r}}{4\pi r} \tag{5}$$

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Consequently, $\mathbf{u}_z^s$ follows from (2) to be

$$\mathbf{u}_z^s(\mathbf{r}) = \frac{A}{\mu k_s^2}(\hat{z}\hat{z}k_s^2 + \nabla_z\nabla) \cdot \hat{a}\frac{e^{ik_sr}}{4\pi r} \tag{6}$$

The z component of (5) characterizes an SH wave while (6) characterizes an SV wave. The P wave can be characterized by θ which has been previously derived to be

$$\begin{aligned}
\theta &= \frac{\nabla\cdot}{\lambda + 2\mu}\int d\mathbf{r}'\frac{e^{ik_c|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}\mathbf{f}(\mathbf{r}') \\
&= \frac{A\nabla\cdot\hat{a}}{\lambda + 2\mu}\frac{e^{ik_cr}}{4\pi r}
\end{aligned} \tag{7}$$

for this particular point source. Alternatively, P wave can be characterized by u_z^p. Using the Sommerfeld identity, the above can be expressed as

$$\Omega_z(\mathbf{r}) = \frac{iA\hat{z}\cdot(\nabla\times\hat{a})}{4\pi\mu}\int_0^\infty dk_\rho\frac{k_\rho}{k_{sz}}e^{ik_{sz}|z|}J_0(k_\rho\rho), \quad \text{SH}, \tag{8a}$$

$$u_z^s(\mathbf{r}) = \frac{iA(\hat{z}\cdot\hat{a}k_s^2 + \frac{\partial}{\partial z}\nabla\cdot\hat{a})}{4\pi\mu k_s^2}\int_0^\infty dk_\rho\frac{k_\rho}{k_{sz}}e^{ik_{sz}|z|}J_0(k_\rho\rho), \quad \text{SV}, \tag{8b}$$

$$u_z^p(\mathbf{r}) = \frac{\pm A\nabla\cdot\hat{a}}{4\pi(\lambda+2\mu)k_c^2}\int_0^\infty dk_\rho k_\rho e^{ik_{cz}|z|}J_0(k_\rho\rho), \quad \text{P} \tag{8c}$$

The above have expressed the field from a point source in terms of a linear superposition of plane waves. We have used u_z^p to characterize P waves so that it has the same dimension as u_z^s.

When the point source is placed above a planarly layered medium, the SH wave characterized by Ω_z will propogate through the layered medium independently of the other waves. Hence, the SH wave for a point source on top of a layered medium can be expressed as

$$\Omega_z(\mathbf{r}) = \frac{iA\hat{z}\cdot(\nabla\times\hat{a})}{4\pi\mu}\int_0^\infty dk_\rho\frac{k_\rho}{k_{sz}}J_0(k_\rho\rho)\left[e^{ik_{sz}|z|} + R_{HH}e^{ik_{sz}(z+2d_1)}\right] \tag{8}$$

Since the SV waves and the P waves are always coupled together in a planarly layered medium, we need to write them as a couplet:

$$\phi = \begin{bmatrix} u_z^s \\ u_z^p \end{bmatrix} = \int_0^\infty dk_\rho k_\rho e^{i\bar{\mathbf{k}}_z|z|}\begin{bmatrix} \widetilde{u}_{z\pm}^s \\ \widetilde{u}_{z\pm}^p \end{bmatrix} \quad \begin{array}{c} z > 0 \\ z < 0 \end{array} \tag{9}$$

where

$$\widetilde{u}_{z\pm}^s = \frac{iA(\hat{z}\cdot\hat{a}k_s^2 \pm ik_{sz}\nabla_\pm^s\cdot\hat{a})}{4\pi\mu k_s^2 k_{sz}}J_0(k_\rho\rho), \quad \widetilde{u}_{z\pm}^p = \frac{\pm A\nabla_\pm^p\cdot\hat{a}}{4\pi(\lambda+2\mu)k_c^2}J_0(k_\rho\rho) \tag{10a}$$

$$\bar{\mathbf{k}}_z = \begin{bmatrix} k_{sz} & 0 \\ 0 & k_{cz} \end{bmatrix}, \nabla_\pm^s = \nabla_s \pm \hat{z}ik_{sz}, \nabla_\pm^p = \nabla_s \pm \hat{z}ik_{cz}. \tag{10b}$$

When the point excitation is placed on top of a layered medium, we have

$$\phi = \int_0^\infty dk_\rho k_\rho\left[e^{i\bar{\mathbf{k}}_z|z|}\cdot\widetilde{\mathbf{u}}_\pm + e^{i\bar{\mathbf{k}}_z(z+d_1)}\cdot\bar{\mathbf{R}}\cdot e^{i\bar{\mathbf{k}}_z d_1}\cdot\widetilde{\mathbf{u}}_-\right]J_0(k_\rho\rho) \tag{10}$$

where $\widetilde{\mathbf{u}}_\pm^t = \left[\widetilde{u}_{z\pm}^s, \widetilde{u}_{z\pm}^p\right]$ and $\bar{\mathbf{R}}$ is the appropriate refecltion matrix describing the reflection and cross-coupling between the SV and P waves.

The above derivation could be repeated with the Weyl identity if we so wish.

Exercise

The shear part of the displacement field $\mathbf{u}^s(\mathbf{r})$ is related to $\mathbf{\Omega}$ as given by (2) and (3). Show that if $\mathbf{u}_z^s$, $\mathbf{\Omega}_z$ are known, and they have plane wave behaviour in the z variable, i.e., $\mathbf{\Omega}_z$, $\mathbf{u}_z^s \sim e^{\pm ik_{sz}z}$, then $\mathbf{u}_s^s$, $\mathbf{\Omega}_s$, the components transverse to z are given by

$$\mathbf{u}_s^s = \frac{1}{k_s^2 - k_{sz}^2}\left[\nabla_s \times \mathbf{\Omega}_z + \nabla_s \frac{\partial}{\partial z}u_z^s\right],$$

$$\mathbf{\Omega}_s^s = \frac{1}{k_s^2 - k_{sz}^2}\left[k_s^2\nabla_s \times \mathbf{u}_z^s + \nabla_s \frac{\partial}{\partial z}\Omega_z\right]$$

In other words, in a homogeneous, isotropic, source free region, all components of $\mathbf{u}^s$ and $\mathbf{\Omega}$ are known if u_z^s and Ω_z are known.

Hint: Write $\mathbf{u}^s = \mathbf{u}_s^s + \mathbf{u}_z^s$, $\mathbf{\Omega} = \mathbf{\Omega}_s + \mathbf{\Omega}_z$, $\nabla = \nabla_s + \hat{z}\frac{\partial}{\partial z}$ in (2) and (3) and equals the transverse to z components.

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Elastic Wave Class Note 6, ECE471, U. of Illinois

W. C. Chew

Fall, 1991

Finite Difference Scheme for the Elastic Wave Equation

The equation of motion for elastic waves is given by

$$\rho\frac{\partial^2 u_x}{\partial t^2} = \frac{\partial \mathcal{T}_{xx}}{\partial x} + \frac{\partial \mathcal{T}_{xz}}{\partial z} \tag{1a}$$

$$\rho\frac{\partial^2 u_z}{\partial t^2} = \frac{\partial \mathcal{T}_{xz}}{\partial x} + \frac{\partial \mathcal{T}_{zz}}{\partial z} \tag{1b}$$

$$\mathcal{T}_{xx} = (\lambda + 2\mu)\frac{\partial u_x}{\partial x} + \lambda\frac{\partial u_z}{\partial z} \tag{1c}$$

$$\mathcal{T}_{zz} = (\lambda + 2\mu)\frac{\partial u_z}{\partial z} + \lambda\frac{\partial u_x}{\partial x} \tag{1d}$$

$$\mathcal{T}_{xz} = \mathcal{T}_{zx} = \mu(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}) \tag{1e}$$

Defining $v_i = \partial u_i/\partial t$, the above can be transformed into a first-order system, i.e.,

$$\frac{\partial v_x}{\partial t} = \rho^{-1}\left(\frac{\partial \mathcal{T}_{xx}}{\partial x} + \frac{\partial \mathcal{T}_{xz}}{\partial z}\right) \tag{2a}$$

$$\frac{\partial u_z}{\partial t} = \rho^{-1}\left(\frac{\partial \mathcal{T}_{xz}}{\partial x} + \frac{\partial \mathcal{T}_{zz}}{\partial z}\right) \tag{2b}$$

$$\frac{\partial \mathcal{T}_{xx}}{\partial t} = (\lambda + 2\mu)\frac{\partial v_x}{\partial x} + \lambda\frac{\partial v_z}{\partial z} \tag{2c}$$

$$\frac{\partial \mathcal{T}_{zz}}{\partial t} = (\lambda + 2\mu)\frac{\partial v_z}{\partial z} + \lambda\frac{\partial v_x}{\partial x} \tag{2d}$$

$$\frac{\partial \mathcal{T}_{xz}}{\partial t} = \mu\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right) \tag{2e}$$

Using central differencing, the above could be written as

$$v_{x,i,j}^{k+\frac{1}{2}} - v_{x,i,j}^{k-\frac{1}{2}} = \rho_{ij}^{-1}\frac{\Delta t}{\Delta x}\left[\mathcal{T}_{xx,i+\frac{1}{2},j}^k - \mathcal{T}_{xx,i-\frac{1}{2},j}^k\right]$$

$$+\rho_{ij}^{-1}\frac{\Delta t}{\Delta z}\left[\mathcal{T}_{xz,i,j+\frac{1}{2}}^k - \mathcal{T}_{xz,i,j-\frac{1}{2}}^k\right] \tag{3a}$$

$$v_{z,i+\frac{1}{2},j+\frac{1}{2}}^{k+\frac{1}{2}} - v_{z,i+\frac{1}{2},j+\frac{1}{2}}^{k-\frac{1}{2}} = \rho_{i+\frac{1}{2},j+\frac{1}{2}}^{-1} \frac{\Delta t}{\Delta x} \left[T_{xz,i+1,j+\frac{1}{2}}^{k} - T_{xz,i,j+\frac{1}{2}}^{k} \right]$$

$$+ \rho_{i+\frac{1}{2},j+\frac{1}{2}}^{-1} \frac{\Delta t}{\Delta z} \left[T_{zz,i+\frac{1}{2},j+1}^{k} - T_{zz,i+\frac{1}{2},j}^{k} \right] \tag{3b}$$

$$T_{xx,i+\frac{1}{2},j}^{k+1} - T_{xx,i+\frac{1}{2},j}^{k} = (\lambda + 2\mu)_{i+\frac{1}{2},j} \frac{\Delta t}{\Delta x} \left[v_{x,i+1,j}^{k+\frac{1}{2}} - v_{x,i,j}^{k+\frac{1}{2}} \right]$$

$$+ \lambda_{i+\frac{1}{2},j} \frac{\Delta t}{\Delta z} \left[v_{z,i+\frac{1}{2},j+\frac{1}{2}}^{k+\frac{1}{2}} - v_{z,i+\frac{1}{2},j-\frac{1}{2}}^{k+\frac{1}{2}} \right] \tag{3c}$$

$$T_{zz,i+\frac{1}{2},j}^{k+1} - T_{zz,i+\frac{1}{2},j}^{k} = (\lambda + 2\mu)_{i+\frac{1}{2},j} \frac{\Delta t}{\Delta z} \cdot \left[v_{z,i,j+1}^{k+\frac{1}{2}} - v_{z,i,j}^{k+\frac{1}{2}} \right]$$

$$+ \lambda_{i+\frac{1}{2},j} \frac{\Delta t}{\Delta x} \left[v_{x,i+\frac{1}{2},j}^{k+\frac{1}{2}} - v_{x,i,j}^{k+\frac{1}{2}} \right] \tag{3d}$$

$$T_{xz,i,j+\frac{1}{2}}^{k+1} - T_{xz,i,j+\frac{1}{2}}^{k} = \mu_{i,j+\frac{1}{2}} \frac{\Delta t}{\Delta z} \left[v_{x,i,j+1}^{k+\frac{1}{2}} - v_{x,i,j}^{k+\frac{1}{2}} \right]$$

$$+ \mu_{i,j+\frac{1}{2}} \frac{\Delta t}{\Delta x} \left[v_{z,i+\frac{1}{2},j+\frac{1}{2}}^{k+\frac{1}{2}} - v_{z,i-\frac{1}{2},j+\frac{1}{2}}^{k+\frac{1}{2}} \right] \tag{3}$$

For a homogenous medium, the stability criterion is

$$v_\rho \Delta t \sqrt{\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta z)^2}} < 1 \tag{4}$$