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Recurrence Relations for Three-Dimensional Scalar Addition Theorem

W. C. Chew

Department of Electrical and Computer Engineering
University of Illinois
Urbana, IL 61801
USA

Abstract–Recurrence relations for the elements of a translation matrix in the scalar addition theorem in three-dimensions using spherical harmonics are derived. These recurrence relations are more efficient to evaluate compared to the use of Gaunt coefficients evaluated with Wigner $3j$ symbols or with recurrence relations. The efficient evaluation of the addition theorem is important in a number of wave scattering calculations including fast recursive algorithms.

I. INTRODUCTION

The use of addition theorems is central to wave scattering theory [1-3]. The development of fast recursive algorithms recently [4-8] using translation formulas further underscores their importance. However, there is a curious point about the addition theorems. In two dimensions, the addition theorem exists in simple forms. The recent use of raising operators [8, p. 62] makes its derivation even simpler. On the other hand, the addition theorem in three dimensions is very complicated except for the monopole term. This fact warrants the further investigation of the addition theorem in three dimensions.

The addition theorem in two dimensions is [8, p. 591]

$$J_m(k\rho)e^{im\phi} = \sum_{n=-\infty}^{\infty} J_{m-n}(k\rho'')J_n(k\rho')e^{in\phi'-i(n-m)\phi''} \tag{1}$$

where $J_n(x)$ is a cylindrical Bessel function. The relation between $\boldsymbol{\rho}$, $\boldsymbol{\rho}'$, and $\boldsymbol{\rho}''$ is shown in Fig. 1.

If the infinite summation in (1) is truncated at N terms, it is seen that the computational effort to calculate (1) is linearly proportional to N. In contrast, the addition theorem in three dimensions is [8-10]

$$Y_{nm}(\theta,\phi)j_n(kr) = \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu} Y_{\nu\mu}(\theta',\phi')j_\nu(kr')\beta_{\nu\mu,nm} \tag{2}$$

where $\beta_{\nu\mu,nm}$ is the element of the translation matrix. It is given by

$$\beta_{\nu\mu,nm} = \sum_p 4\pi i^{(\nu+p-n)}Y_{p,m-\mu}(\theta'',\phi'')j_p(kr'')A(m,n,-\mu,\nu,p) \tag{3}$$

where

$$A(m,n,-\mu,\nu,p) = (-1)^m[(2n+1)(2\nu+1)(2p+1)/4\pi]^{\frac{1}{2}} \cdot \begin{pmatrix} n & \nu & p \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} n & \nu & p \\ -m & \mu & m-\mu \end{pmatrix} \tag{3a}$$

and

$$Y_{nm}(\theta,\phi) = (-1)^m \left[\frac{(n-m)!}{(n+m)!}\frac{2n+1}{4\pi}\right]^{\frac{1}{2}} P_n^m(\cos\theta)e^{im\phi}, \tag{3b}$$

with the property that $Y_{n,-m}(\theta,\phi) = (-1)^m Y_{nm}^*(\theta,\phi)$, and $j_n(x)$ is a spherical Bessel function, and $\begin{pmatrix} n & \nu & p \\ m & \mu & q \end{pmatrix}$ is the Wigner $3j$ symbol. $A(m,n,-\mu,\nu,p)$ is related to the Gaunt coefficients [1, 9]. The relation between $\mathbf{r}$, $\mathbf{r}'$, and $\mathbf{r}''$ is shown in Fig. 1. In (2), it is seen that $(\nu,\mu) \in \mathbf{S}$ where $\mathbf{S}$ is a set defined by $\mathbf{S} = \{(\nu,\mu) \mid 0 \le \nu \le \infty, -\nu \le \mu \le \nu, \nu \in \mathbf{Z}, \mu \in \mathbf{Z}\}$, where $\mathbf{Z}$ is the set of integer numbers. Similarly, $(m,n) \in \mathbf{S}$ also in (2).

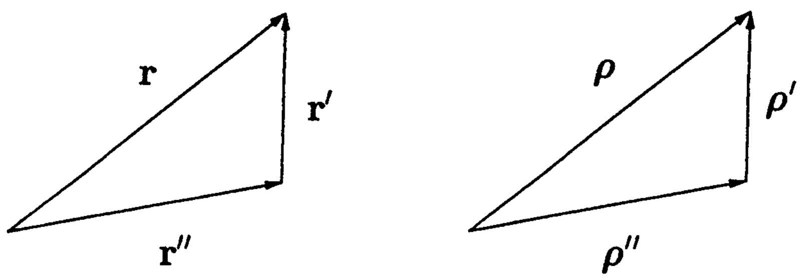

Figure 1. The relationships between $\mathbf{r}$, $\mathbf{r}'$, and $\mathbf{r}''$; and $\boldsymbol{\rho}$, $\boldsymbol{\rho}'$, and $\boldsymbol{\rho}''$ for the addition theorems in three and two dimensions respectively.

For real k, it can be shown that $\beta_{\nu\mu,nm}$ is a unitary matrix, i.e.

$$\sum_{\nu\mu} \beta_{n'm',\nu\mu}^* \beta_{\nu\mu,nm} = \delta_{nm,n'm'} \tag{4}$$

or $\overline{\boldsymbol{\beta}}^{-1} = \overline{\boldsymbol{\beta}}^{\dagger}$ where $\overline{\boldsymbol{\beta}}$ is the translation matrix.

Furthermore, it can be shown that for real k,

$$\beta_{nm,\nu\mu} = (-1)^{n-\nu}\beta_{\nu\mu,nm}^* \tag{5}$$

Since translating in the negative $\mathbf{r}''$ direction is the inverse of translating in the positive $\mathbf{r}''$ direction, it follows that

$$\overline{\boldsymbol{\beta}}(\pi-\theta'',\phi''+\pi) = \overline{\boldsymbol{\beta}}^{-1}(\theta'',\phi'') = \overline{\boldsymbol{\beta}}^{\dagger}(\theta'',\phi'') \tag{6}$$

Note that if only N terms are kept in the summation over ν, the outer two summations over ν and μ will result in $(N+1)^2 \sim O(N^2)$ terms. If n in (2) is such that $n = 0, 1, \cdots, N'$, and $-n < m < n$, then there will be $O(N'^2)$ terms over the mn indices. Consequently, the dimension of the translation matrix $\beta_{\nu\mu,nm}$ is $O(N^2) \times O(N'^2)$. But the calculation of $\beta_{\nu\mu,nm}$ involves a summation over p as shown in (3). Because the Wigner $3j$ symbols are only nonzero when $n+\nu \geq p \geq |n-\nu|$, the number of terms in the summation over p is proportional to n. Consequently, the computer time needed to calculate the $\beta_{\nu\mu,nm}$ matrix is $O(N'^3 N^2)$. Moreover, the calculation of the Wigner $3j$ symbols involves a large number of factorials. Hence, a large amount of effort is expended in calculating the Gaunt coefficients.

In order to reduce the computational effort for finding the Gaunt coefficients, recurrence relations have been derived for them [1, 11]. We shall call these recurrence relations the Bruning-Lo-Fuller recurrence relations. However, if the translation is not in the z-direction, the recurrence relations are rather complex, and the reduction of computer time by a small factor is observed [12]. Even though the Bruning-Lo-Fuller recurrence relations expedite the calculation of the Gaunt coefficients, the computational complexity of calculating the $\beta_{\nu\mu,nm}$ matrix is not reduced since the summation over p in (3) is not removed.

In view of this, a more efficient way of calculating the translation matrix $\beta_{\nu\mu,nm}$ is needed. Hence, recurrence relations expressed more directly in terms of $\beta_{\nu\mu,nm}$ would be more expedient. Such recurrence relations shall be derived. As shall be shown, the calculation of the addition theorem using such recurrence relations will reduce its computation complexity. The initial value for the recursion will make use of the addition theorem for the spherically symmetric harmonic, which has the following simple form [8, p. 194 or p. 594],

$$Y_{00}(\theta, \phi) j_0(kr) = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} Y_{\nu\mu}(\theta', \phi') j_\nu(kr') Y_{\nu,-\mu}(\theta'', \phi'') j_\nu(kr'') \sqrt{4\pi} (-1)^{\mu+\nu} \tag{7}$$

Hence,

$$\beta_{\nu\mu,oo} = (-1)^{\mu+\nu} \sqrt{4\pi} Y_{\nu,-\mu}(\theta'', \phi'') j_\nu(kr'') \tag{7a}$$

is known in a very simple form for $(\nu, \mu) \in \mathbf{S}$. Notice that $\beta_{\nu,-\mu,oo} = (-1)^\mu \beta^*_{\nu\mu,oo}$ if k is real†, and $\beta_{\nu\mu,oo}$ could be calculated recursively because $P_\nu^\mu(\cos\theta'')$ and $j_\nu(kr'')$ could be calculated recursively.

II. DERIVATION OF USEFUL FORMULAS

Before deriving recurrence relations for the scalar addition theorem, it is useful to establish some important formulas from which the recurrence relations could be derived. First, using the fact that* [13, 14]

$$\frac{\partial}{\partial r} j_n(r) = -j_{n+1}(r) + \frac{n}{r} j_n(r) = j_{n-1}(r) - \frac{n+1}{r} j_n(r) \tag{8a}$$

† When k is complex, the relation still holds if the conjugation implies that only $e^{-i\mu\phi''}$ in $Y_{\nu,-\mu}$ is conjugated.
* r is normalized here so that $k = 1$.

$$\frac{\partial}{\partial\theta}P_n^m(\cos\theta) = \frac{m\cos\theta}{\sin\theta}P_n^m(\cos\theta) - P_n^{m+1}(\cos\theta) \tag{8b}$$

it can be shown that

$$\frac{\partial}{\partial z}j_n(r)P_n^m(\cos\theta) = -\cos\theta P_n^m(\cos\theta)j_{n+1}(r) + \frac{j_n(r)}{r}[(n-m)\cos\theta P_n^m(\cos\theta) + \sin\theta P_n^{m+1}(\cos\theta)] \tag{9}$$

It can also be shown that†

$$(n+m)P_{n-1}^m(\cos\theta) = (n-m)\cos\theta P_n^m(\cos\theta) + \sin\theta P_n^{m+1}(\cos\theta) \tag{10a}$$

$$\cos\theta P_n^m(\cos\theta) = \frac{1}{2n+1}[(n-m+1)P_{n+1}^m(\cos\theta) + (n+m)P_{n-1}^m(\cos\theta)] \tag{10b}$$

Using (10a), (10b), and (8a), (9) could be simplified to

$$\frac{\partial}{\partial z}j_n(r)P_n^m(\cos\theta) = -\frac{n-m+1}{2n+1}j_{n+1}(r)P_{n+1}^m(\cos\theta) + \frac{n+m}{2n+1}j_{n-1}(r)P_{n-1}^m(\cos\theta) \tag{11}$$

where $P_{n-1}^m(\cos\theta) = 0$ when $m > n-1$. With the definition of $Y_{nm}(\theta,\phi)$ given by (3b), it can be shown from (11) that

$$\frac{\partial}{\partial z}j_n(r)Y_{nm}(\theta,\phi) = -\left[\frac{(n+m+1)(n-m+1)}{(2n+1)(2n+3)}\right]^{\frac{1}{2}} j_{n+1}(r)Y_{n+1,m}(\theta,\phi) + \left[\frac{(n+m)(n-m)}{(2n+1)(2n-1)}\right]^{\frac{1}{2}} j_{n-1}(r)Y_{n-1,m}(\theta,\phi) \tag{12}$$

In other words,

$$\frac{\partial}{\partial z}\psi_{nm}(\mathbf{r}) = a_{n,m}^+\psi_{n+1,m}(\mathbf{r}) + a_{nm}^-\psi_{n-1,m}(\mathbf{r}) \tag{13}$$

where

$$\psi_{nm}(\mathbf{r}) = j_n(r)Y_{nm}(\theta,\phi) \tag{13a}$$

$$a_{nm}^+ = -\left[\frac{(n+m+1)(n-m+1)}{(2n+1)(2n+3)}\right]^{\frac{1}{2}} \tag{13b}$$

$$a_{nm}^- = \left[\frac{(n+m)(n-m)}{(2n+1)(2n-1)}\right]^{\frac{1}{2}} \tag{13c}$$

Equation (13) implies that $\frac{\partial}{\partial z}$ operator, when operating on $\psi_{nm}(\mathbf{r})$, generates a higher and a lower order multipole without affecting m.

† Note that many recurrence relations for Lengendre functions in [13] are incorrect, but those in [14, p. 401] are correct.

Next, an operator on ψ_{nm} that will change the values of m needs to be found. To do this, we define a circulating operator*

$$C_+ = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \tag{14}$$

It could be easily shown that in spherical coordinates

$$C_+ = e^{i\phi}\left[\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta} + \frac{i}{r\sin\theta}\frac{\partial}{\partial\phi}\right] \tag{15}$$

Then, using (8a) and (8b), it can be shown that

$$C_+ j_n(r)P_n^m(\cos\theta)e^{im\phi} = e^{i(m+1)\phi}\left\{\sin\theta P_n^m(\cos\theta)\left[j_{n-1}(r) - \frac{n+1}{r}j_n(r)\right] - \frac{\cos\theta}{r}j_n(r)P_n^{m+1}(\cos\theta) - \frac{m\sin\theta}{r}j_n(r)P_n^m(\cos\theta)\right\} \tag{16}$$

Using the fact that

$$\sin\theta P_n^m(\cos\theta) = \frac{1}{2n+1}[P_{n+1}^{m+1}(\cos\theta) - P_{n-1}^{m+1}(\cos\theta)] \tag{17}$$

and (10b) for $m \to m+1$, we have

$$C_+ j_n(r)P_n^m(\cos\theta)e^{im\phi} = -\frac{e^{i(m+1)\phi}}{2n+1}[j_{n-1}(r)P_{n-1}^{m+1}(\cos\theta) + j_{n+1}(r)P_{n+1}^{m+1}(\cos\theta)] \tag{18a}$$

Consequently, we can show that

$$C_+\psi_{nm}(\mathbf{r}) = b_{nm}^-\psi_{n-1,m+1}(\mathbf{r}) + b_{nm}^+\psi_{n+1,m+1}(\mathbf{r}) \tag{18b}$$

where

$$b_{nm}^- = \left[\frac{(n-m)(n-m-1)}{(2n+1)(2n-1)}\right]^{\frac{1}{2}}, \qquad b_{nm}^+ = \left[\frac{(n+m+2)(n+m+1)}{(2n+1)(2n+3)}\right]^{\frac{1}{2}} \tag{18c}$$

The above formulas (13) and (18b) will be used to derive the recurrence relations for the addition theorem.

III. THE RECURRENCE RELATIONS

An addition theorem for spherical harmonics can be written as

$$\psi_{nm}(\mathbf{r}) = \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}\psi_{\nu\mu}(\mathbf{r}')\beta_{\nu\mu,nm} \tag{19}$$

where ν is summed from 0 to ∞, while μ is summed from $-\nu$ to ν. Since $\frac{\partial}{\partial z} = \frac{\partial}{\partial z'}$, operating both sides of (19) by $\frac{\partial}{\partial z}$ yields

$$a_{nm}^+\psi_{n+1,m}(\mathbf{r}) = -a_{nm}^-\psi_{n-1,m}(\mathbf{r}) + \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}a_{\nu\mu}^+\psi_{\nu+1,\mu}(\mathbf{r}')\beta_{\nu\mu,nm} + \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}a_{\nu\mu}^-\psi_{\nu-1,\mu}(\mathbf{r}')\beta_{\nu\mu,nm} \tag{20}$$

* This is the same as the raising operator in [8, p. 62].

By changing $\nu+1 \rightarrow \nu$ in the first sum, and $\nu-1 \rightarrow \nu$ in the second sum, we have

$$a_{nm}^{+}\psi_{n+1,m}(\mathbf{r}) = -\, a_{nm}^{-}\psi_{n-1,m}(\mathbf{r}) + \sum_{\nu=1}^{\infty} \sum_{\mu=-(\nu-1)}^{\nu-1} \psi_{\nu\mu}(\mathbf{r}')a_{\nu-1,\mu}^{+}\beta_{\nu-1,\mu,nm} + \sum_{\nu=-1}^{\infty} \sum_{\mu=-(\nu+1)}^{\nu+1} \psi_{\nu\mu}(\mathbf{r}')a_{\nu+1,\mu}^{-}\beta_{\nu+1,\mu,nm} \tag{21}$$

By noticing that $a_{-1,0}^{+} = 0$, $a_{\nu-1,\pm\nu}^{+} = 0$, $a_{0,0}^{-} = 0$, and $a_{\nu+1,\pm(\nu+1)}^{-} = 0$, the limits of the summations in (21) can be rewritten as

$$a_{nm}^{+}\psi_{n+1,m}(\mathbf{r}) = -a_{nm}^{-}\psi_{n-1,m}(\mathbf{r}) + \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} \psi_{\nu\mu}(\mathbf{r}') \cdot [a_{\nu-1,\mu}^{+}\beta_{\nu-1,\mu,nm} + a_{\nu+1,\mu}^{-}\beta_{\nu+1,\mu,nm}] \tag{22}$$

Using (19) to expand $\psi_{n+1,m}(\mathbf{r})$ and $\psi_{n-1,m}(\mathbf{r})$ as well and equating terms in the series, we have

$$a_{nm}^{+}\beta_{\nu\mu,n+1,m} = -a_{nm}^{-}\beta_{\nu\mu,n-1,m} + a_{\nu-1,\mu}^{+}\beta_{\nu-1,\mu,nm} + a_{\nu+1,\mu}^{-}\beta_{\nu+1,\mu,nm} \tag{23}$$

Equation (23) allows one to find $\beta_{\nu\mu,n+1,m}$ for $(\nu,\mu) \in \mathbf{S}$ if $\beta_{\nu\mu,n-1,m}$ and $\beta_{\nu\mu,nm}$ are known for $(\nu,\mu) \in \mathbf{S}$. Hence, given the addition theorem for the $(n-1,m)$ and (n,m) harmonics, that for $(n+1,m)$ harmonics could be found. Notice that $a_{mm}^{-} = 0$ so that the (n,m) harmonic alone can be used to determine the $(n+1,m)$ harmonic when $n = m$. Hence, the initial value for the recurrence relation (23) can be obtained from $\beta_{\nu\mu,nm}$ if they are known.

The recurrence relation in (23) does not cycle in m. Therefore, another recurrence relation is needed if the addition theorem for different m values are needed as well. To do this, we apply the C_{+} operator on (19). Note that the C_{+} operator is also coordinate independent so that it has the same effect on $\psi_{nm}(\mathbf{r})$ and $\psi_{\nu\mu}(\mathbf{r}')$. Consequently, we have

$$b_{nm}^{+}\psi_{n+1,m+1}(\mathbf{r}) + b_{nm}^{-}\psi_{n-1,m+1}(\mathbf{r}) = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} b_{\nu\mu}^{+}\psi_{\nu+1,\mu+1}(\mathbf{r}')\beta_{\nu\mu,nm} + \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} b_{\nu\mu}^{-}\psi_{\nu-1\mu+1}(\mathbf{r}')\beta_{\nu\mu,nm} \tag{24}$$

By letting $\nu+1 \rightarrow \nu$, and $\mu+1 \rightarrow \mu$ in the first term and $\nu-1 \rightarrow \nu$, and $\mu+1 \rightarrow \mu$ in the second term in (24), we have

$$b_{nm}^{+}\psi_{n+1,m+1}(\mathbf{r}) + b_{nm}^{-}\psi_{n-1,m+1}(\mathbf{r}) = \sum_{\nu=1}^{\infty} \sum_{\mu=-\nu+2}^{\nu} \psi_{\nu\mu}(\mathbf{r}')b_{\nu-1,\mu-1}^{+}\beta_{\nu-1,\mu-1,nm} + \sum_{\nu=-1}^{\infty} \sum_{\mu=-\nu}^{\nu+2} \psi_{\nu\mu}(\mathbf{r}')b_{\nu+1,\mu-1}^{-}\beta_{\nu+1,\mu-1,nm} \tag{24a}$$

Using arguments similar to that for (21), the summation indices could be rearranged so that

$$b_{nm}^{+}\psi_{n+1,m+1}(\mathbf{r}) = -\, b_{nm}^{-}\psi_{n-1,m+1}(\mathbf{r}) + \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu} \psi_{\nu\mu}(\mathbf{r}')$$

$$[b_{\nu-1,\mu-1}^{+}\beta_{\nu-1,\mu-1,nm} + b_{\nu+1,\mu-1}^{-}\beta_{\nu+1,\mu-1,nm}] \qquad (25)$$

By the same token as (23), a recurrence relation can be derived such that

$$b_{nm}^{+}\beta_{\nu\mu,n+1,m+1} = -\, b_{nm}^{-}\beta_{\nu\mu,n-1,m+1} + b_{\nu-1,\mu-1}^{+}\beta_{\nu-1,\mu-1,nm}$$

$$+\, b_{\nu+1,\mu-1}^{-}\beta_{\nu+1,\mu-1,nm} \qquad (26)$$

Equation (26) allows the addition theorem for $(n+1, m+1)$ harmonic to be found if the addition theorems for $(n-1, m+1)$ and (n, m) harmonics are known. If we let $m = n$ in (26), then $b_{nn}^{-} = 0$, and (26) becomes

$$b_{nn}^{+}\beta_{\nu\mu,n+1,n+1} = b_{\nu-1,\mu-1}^{+}\beta_{\nu-1,\mu-1,nn} + b_{\nu+1,\mu-1}^{-}\beta_{\nu+1,\mu-1,nn} \qquad (27)$$

Hence, given $\beta_{\nu\mu,nn}$ for $(\nu,\mu) \in \mathbf{S}$, $\beta_{\nu\mu,n+1,n+1}$ can be found for $(\nu,\mu) \in \mathbf{S}$. In the discrete space of nm shown in Fig. 2, the coefficients $\beta_{\nu\mu,nm}$ are defined only for $|m| \le n$. They are zero otherwise. Furthermore, using the property of $Y_{n,-m}(\theta,\phi)$ immediately following (3b), it can be shown that

$$\beta_{\nu\mu,n,-m} = (-1)^{\mu+m}\beta_{\nu,-\mu,nm}^{*} \qquad (28)$$

Now, if the addition theorem is needed for the harmonic at A, then (27) can be first used to derive the addition theorem of the harmonic that corresponds to B from the $\beta_{\nu\mu,oo}$ values. Subsequently, (23) could be used to derive the coefficients corresponding to A from B. Equation (28) can then be used to derive $\beta_{\nu\mu,n,-m}$ for $(\nu,\mu) \in S$. Other ways of using the recursion formulas (23), (26) and (27) are possible.

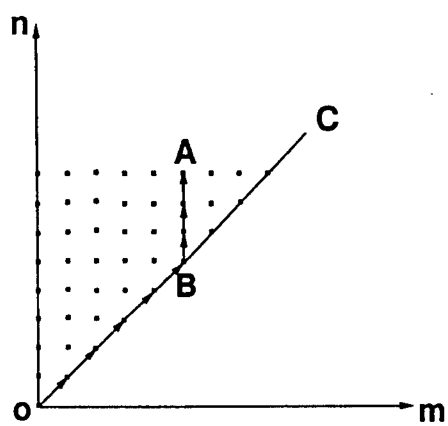

Figure 2. The translation matrix element $\beta_{\nu\mu,nm}$ at A can be found from $\beta_{\nu\mu,oo}$ at O. The recurrence relations can be used first to yield the value of $\beta_{\nu\mu,nm}$ at B where $n = m$. Then the value of $\beta_{\nu\mu,nm}$ at A can be derived from that at B using the recurrence relations.

Equation (23) is like a finite difference equation where the values of $\beta_{\nu\mu,n+1,m}$ are obtained from $\beta_{\nu\mu,n-1,m}$, $\beta_{\nu-1,\mu,nm}$, and $\beta_{\nu+1,\mu,nm}$. Similarly, in (27), $\beta_{\nu\mu,n+1,n+1}$ is determined from $\beta_{\nu-1,\mu-1,nm}$ and $\beta_{\nu+1,\mu-1,nn}$. Hence, starting with the initial values $\beta_{\nu\mu,oo}$ for $0 \le \nu \le N$, the $\beta_{\nu\mu,nm}$ constructed reduces in the length over ν when n is increasing. Therefore, if the translation matrix $\beta_{\nu\mu,nm}$ is needed for $0 \le \nu \le N$, and $0 \le n \le N'$, the initial values on the ν-axis as shown in Fig. 3a where $0 \le \nu \le N + N'$. (Note that in Fig. 3, each point of ν implicitly embeds a range of $-\nu \le \mu \le \nu$. The same statement applies to each point of n.) Extraneous elements of $\beta_{\nu\mu,nm}$ are generated over the domain S when $\beta_{\nu\mu,nm}$ is desired over the domain A.

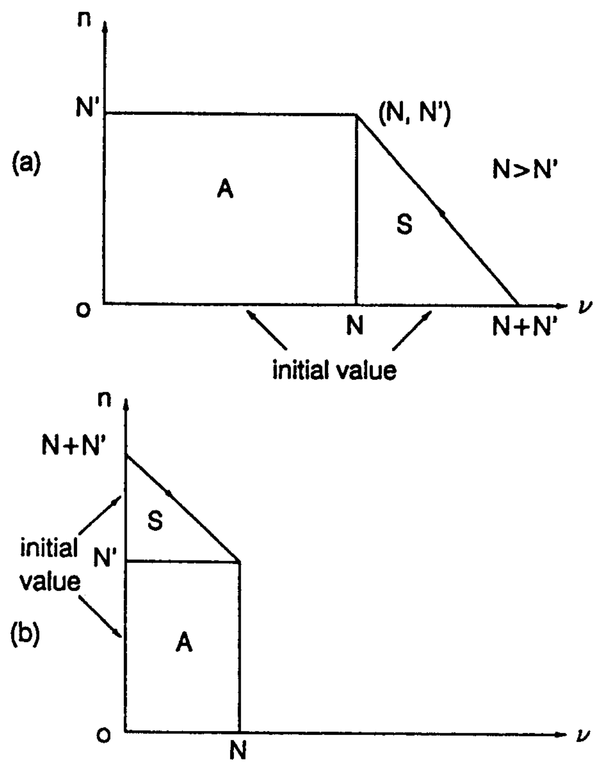

Figure 3. The recurrence relations work like an initial-value problem.

(a) If $\beta_{\nu\mu,nm}$ is required in A where $0 \le \nu \le N$, $0 \le n \le N'$, then the initial value $\beta_{\nu\mu,oo}$ for $0 \le \nu \le N + N'$ is required on the $n = 0$ line.

(b) Alternatively, the values of $\beta_{\nu\mu,nm}$ in A can also be derived from the initial values at $\nu = 0$ line for $0 \le n \le N + N'$. Extraneous values of $\beta_{\nu\mu,nm}$ are generated over S. When $N > N'$, it is more efficient to use case (a), and when $N < N'$, it is more efficient to use case (b), so that the area of S does not become exorbitantly large.

When $N' \gg N$, the domain S could be larger than the domain A, and hence, the method outlined used to generate the translation could be very inefficient. In this case it will be expedient to use the initial values on the n-axis. It can be shown from (5) and (7a) that

$$\beta_{oo,nm} = (-1)^n \beta^*_{nm,oo} = (-1)^n \sqrt{4\pi} Y^*_{nm}(\theta'', \phi'') j_n(kr'') \tag{29}$$

This could be used on the n-axis as initial values. A recurrence relation useful for this mode of application can be obtained from (26) by setting $\mu = -\nu$, yielding

$$b^+_{nm}\beta_{\nu,-\nu,n+1,m+1} + b^-_{nm}\beta_{\nu,-\nu,n-1,m+1} = b^-_{\nu+1,-(\nu+1)}, \beta_{\nu+1,-(\nu+1),nm} \tag{30}$$

The case where initial values are given on the n-axis is shown in Fig. 3.

Note that the number of steps required to find the coefficients $\beta_{\nu\mu,nm}$ is always equal to n and is independent of m. Furthermore, if the $\beta_{\nu\mu,nm}$ for a given n', and $-n' < m < n'$ is found, all the values of $\beta_{\nu\mu,nm}$ for $n < n'$ are also found. No large number of factorials is required in these recurrence relations. Therefore, the computational complexity of finding $\beta_{\nu\mu,nm}$, the translation matrix, is reduced, and the total computer time needed to construct $\beta_{\nu\mu,nm}$ is greatly reduced compared to that of (2). Now, if $\nu = 0, 1, \cdots, N$ and $n = 0, 1, \cdots, N'$ so that there are $(N+1)^2$ terms corresponding to the $\nu\mu$ index, and $(N'+1)^2$ terms corresponding to the nm index, then the computer time needed to compute the $\beta_{\nu\mu,nm}$ translation matrix is proportional to $N^2 N'^2$. This is of reduced complexity compared to calculating this translation matrix using the Gaunt coefficients.

The same recurrence relations can also be used to construct $\alpha_{\nu\mu,nm}$ from $\alpha_{\nu\mu,oo}$, where $\alpha_{\nu\mu,nm}$ is the translation matrix for spherical Hankel functions defined in [8, p. 594]. The above recurrence relations have also been verified by a numerical implementation and comparing with the method using Gaunt coefficients. The recurrence relations give the same result as the method using Gaunt coefficients, but is much more efficient when N is large. For computing a $\overline{\beta}$ matrix which is 25×25, the increase in speed could be as much as 100 times, and the difference is more significant when N becomes large.

From (3), it is seen that when $|n - \nu| \gg kr''$, the value of $\beta_{\nu\mu,nm}$ becomes very small. In this case, the result from the recursions becomes unstable just as that for Bessel functions [13, p. 385]. In this case, the initial value on the ν axis could be used to calculate all the elements for which $n < \nu$, and vice versa for the initial value on the n axis.

IV. CONCLUSIONS

Efficient recurrence relations for the elements of the translation matrix are derived. These recurrence relations are much more efficient than the previous ones. The efficient evaluation of the addition theorem is important in a number of scattering applications, especially in recursive algorithms.

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Weng Cho Chew received the B.S. degree in 1976, both the M.S. and and Engineer's degrees in 1978, and the Ph.D. degree in 1980, all in electrical engineering from the Massachusetts Institute of Technology, Cambridge. His research interest is in the area of waves in inhomogeneous media for various sensing applications, integrated circuits, and microstrip antenna applications. He is now teaching at the University of Illinois. Before coming to the University of Illinois, he was a department manager at Schlumberger-Doll Research. He is an NSF Presidential Young Investigator and has been active with various journals and societies. He has authored a book titled *Waves and Fields in Inhomogeneous Media*.